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On the quantisation of points

Christopher J. Mulvey[a,*], Joan Wick Pelletier[b]

[a] *School of Mathematical Sciences, University of Sussex, Falmer, Brighton, BN1 9QH, UK*

[b] *Department of Mathematics and Statistics, York University, North York, Ont., M3J 1P3, Canada*

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Abstract

In the study of quantales arising naturally in the context of C*-algebras, Gelfand quantales have emerged as providing the basic setting. In this paper, the problem of defining the concept of point of the spectrum Max A of a C*-algebra A, which is one of the motivating examples of a Gelfand quantale, is considered. Intuitively, one feels that points should correspond to irreducible representations of A. Classically, the notions of topological and algebraic irreducibility of a representation are equivalent. In terms of quantales, the irreducible representations of a C*-algebra A are shown to be captured by the notion of an algebraically irreducible representation of the Gelfand quantale Max A on an atomic orthocomplemented sup-lattice S, defined in terms of a homomorphism of Gelfand quantales to the Hilbert quantale $\mathcal{Q}(S)$ of sup-preserving endomorphisms on S. This characterisation leads to a concept of point of an arbitrary Gelfand quantale Q as a map of Gelfand quantales into a Hilbert quantale $\mathcal{Q}(S)$, the inverse image homomorphism of which is an algebraically irreducible representation of Q on the atomic orthocomplemented sup-lattice S. The aptness of this definition of point is demonstrated by observing that in the case of locales it is exactly the classical notion of point, while the Hilbert quantale $\mathcal{Q}(S)$ of an atomic orthocomplemented sup-lattice S has, up to equivalence, exactly one point. In this sense, the Hilbert quantale $\mathcal{Q}(S)$ is considered to be a quantised version of the one-point space.

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1. Introduction

The concept of a quantale was introduced [6] in order to allow the extension [7] of the Gelfand–Naimark representation to non-commutative C*-algebras. The con-

* Corresponding author.

E-mail address: c.j.mulvey@sussex.ac.uk (C.J. Mulvey).

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text within which this extension could be obtained was that of a description of the spectrum Max A of a C*-algebra A, considered as an abstraction of the notion of a non-commutative space. The aim of this paper is to examine the question of defining the concept of a point of a quantale, more specifically of the Gelfand quantales obtained by taking the spectrum of a C*-algebra. In the case of the spectrum Max A of a C*-algebra A, the intuitive feeling is that the points should be the irreducible representations, or perhaps the equivalence classes of irreducible representations, of the C*-algebra A. Indeed, the relevance of the concept of quantale to finding a spectrum for a non-commutative C*-algebra A is exactly that we need to find a non-commutative generalisation of the notion of topological space in order to bring together the irreducible representations of A into an, albeit non-commutative, topological space [6].

To this intuitive idea of the points of the spectrum Max A, we may bring another observation. In the case of a commutative C*-algebra A, the spectrum Max A, considered as a locale, the analogue of a quantale within the commutative context, may be constructed [2] by taking the propositional geometric theory of closed prime ideals of the C*-algebra A. The spectrum is then the Lindenbaum algebra of this theory: that is, the lattice of propositions of the theory, ordered by provable entailment in the theory within constructive logic. This method for the construction of locales, generalising constructively classically defined topological spaces, has proved valuable in a number of situations [1,9,11]. The points of the locale then correspond exactly to the classical models of the theory, hence in that case to the maximal ideals of the commutative C*-algebra.

The identification of the points of the spectrum Max A of a not necessarily commutative C*-algebra A might therefore be effected by examining the extent to which its spectrum may be constructed by considering within quantal logic, by which we shall mean an appropriate non-commutative analogue of constructive logic, an adaptation of the theory considered in the commutative case. The points of the spectrum should then correspond to the models within quantal logic of the theory considered. The problem of defining a concept of a point of a quantale thereby becomes that of defining what is meant by a classical model of a propositional theory within quantal logic. In turn, this involves investigating which quantales should be considered within this logic to take the place of the locale Ω of subsets of the singleton set $\mathbf{1}$ within constructive logic. Intuitively, these may be expected to be quantales which reflect localically to the locale Ω, yet which also have some intrinsically quantalic aspects.

This approach to considering what should be meant by the concept of a point of a quantale appears to be productive. The fact that irreducible representations of a C*-algebra A may be expected to yield the points of the spectrum Max A, and hence the classical models of any propositional geometric theory within quantal logic which generates the spectrum, leads one to identify the quantales which provide this generalisation of the locale Ω in the context of quantales, and the maps of quantales which generalise the points of a locale. In particular, this provides evidence for what may be an appropriate concept of a quantal space, and indicates that the spectrum of a C*-algebra may indeed be expected to be spatial in this particular sense.

2. The spectrum of a C*-algebra

In this section we recall the description of the spectrum $\mathrm{Max}\,A$ of a C*-algebra A, together with the properties of the spectrum which will be needed in what follows. The C*-algebra A considered will be assumed throughout to be unital. We recall first the definition of the concept of quantale [6], on which the definition of the spectrum of a C*-algebra is based:

Definition 2.1. By a *quantale* Q is meant a lattice having arbitrary joins $\bigvee$ together with an associative product & satisfying

$$a\&\bigvee b_i = \bigvee a\&b_i$$

and

$$\bigvee a_i\&b = \bigvee a_i\&b$$

for all $a, b, a_i, b_i \in Q$. The quantale Q is said to be *unital* provided that there exists an element $e \in Q$ for which

$$e\&a = a = a\&e$$

for all $a \in Q$.

The intention here is to abstract the lattice of open subsets of a not necessarily commutative topological space, of which the spectrum of a C*-algebra A is the motivating example. The operation & should be considered to represent that of not necessarily commutative intersection of open subsets. It is evident that in the case that the operation & is the meet of the lattice, then these axioms describe exactly the concept of a locale. The principle being exploited in extending the concept of spectrum [7] is that considering non-commutative C*-algebras requires the introduction of the concept of a non-commutative topological space.

Definition 2.2. By the *spectrum* $\mathrm{Max}\,A$ of a C*-algebra A is meant the quantale of closed linear subspaces of A, together with the product & which is defined by setting

$$M\&N = \overline{M \cdot N}$$

to be the closure of the product of linear subspaces for each $M, N \in \mathrm{Max}\,A$. The join of the lattice is, of course, given by taking

$$\bigvee_i M_i = \overline{\sum_i M_i}$$

to be the closure of the sum of linear subspaces for each family $M_i \in \mathrm{Max}\,A$. The spectrum is a unital quantale, with unit given by the closed linear subspace generated by the unit of the C*-algebra A.

It may be remarked that the construction of the spectrum is dually functorial with respect to a definition of the category of quantales which provides a non-commutative counterpart of that of the category of locales. Explicitly, we recall the following:

Definition 2.3. By a *map* $\varphi : Q \to Q'$ *of quantales* is meant a mapping

$$\varphi^* : Q' \to Q,$$

referred to as the *inverse image homomorphism* from the quantale Q' to the quantale Q, which preserves the operations of product & and of arbitrary join $\bigvee$ of the quantales. The map is said to be *unital* provided that the quantales Q and Q' are unital and that

$$e \leq \varphi^*(e')$$

for $e \in Q$, $e' \in Q'$ respectively the units of Q and Q'.

The category of quantales is then that of quantales and of maps of quantales, with composition of maps given by composition of inverse image homomorphisms and with identity maps given by identity inverse image homomorphisms. With respect to these definitions, one has the following:

Theorem 2.1. *The spectrum* $\mathrm{Max}\, A$ *of a* C*-*algebra* A *determines a functor*

$$\mathrm{Max} : (\mathrm{C}^*\text{-}\mathtt{Algebras})^{op} \to \mathtt{Quantales}$$

from the dual of the category of C*-*algebras to the category of quantales.*

Proof. The map of quantales $\mathrm{Max}\,\varphi : \mathrm{Max}\, A' \to \mathrm{Max}\, A$ from the spectrum of the C*-algebra A' to the spectrum of the C*-algebra A determined by a map of C*-algebras $\varphi : A \to A'$ is that of which the inverse image homomorphism assigns to each closed linear subspace $M \in \mathrm{Max}\, A$ of the C*-algebra A the closure $\overline{\varphi(M)} \in \mathrm{Max}\, A'$ of its image in the C*-algebra A'. □

In the case that the map $\varphi : A \to A'$ of C*-algebras is unital, then the map $\mathrm{Max}\varphi : \mathrm{Max}\, A' \to \mathrm{Max}\, A$ of quantales is unital. The functor therefore restricts canonically to a functor from the dual of the category of unital C*-algebras to the category of unital quantales. It may, in fact, be remarked that the map $\mathrm{Max}\varphi : \mathrm{Max}\, A' \to \mathrm{Max}\, A$ of quantales is in this case such that the inverse image of the unit of the quantale $\mathrm{Max}\, A$ is mapped to, rather than just above, the unit of the quantale $\mathrm{Max}\, A'$. The reason that this stricter condition is not required of a map of unital quantales in general will become apparent later.

3. Gelfand quantales

The spectrum $\mathrm{Max}\, A$ inherits from the C*-algebra A additional structure and properties which make it more amenable than an arbitrary quantale [8]. To begin with, the spectrum $\mathrm{Max}\, A$ is an involutive quantale, in the following sense:

Definition 3.1. By an *involutive quantale* Q is meant a quantale together with an involution $*$ satisfying the conditions that

$$a^{**} = a,$$

$$(a\&b)^* = b^*\&a^*,$$

$$\text{and} \quad \left(\bigvee_i a_i\right)^* = \bigvee_i a_i^*,$$

for all $a, b, a_i \in Q$.

The involution on the spectrum $\operatorname{Max} A$ of a C*-algebra A is that which assigns to any closed linear subspace M the adjoint closed linear subspace

$$M^* = \{a^* \in A \mid a \in M\}.$$

It may be remarked at this point that in any unital involutive quantale Q, the unit $e \in Q$ is *self-adjoint*, that is to say, satisfies the condition that

$$e = e^*.$$

It may also be noted that any locale may trivially be made into an involutive quantale by giving it the involution which maps each element to itself. For an arbitrary quantale, the identity mapping does not determine an involution in the above sense, since an involution has to reverse the order of any product in the quantale.

Equally, the map of quantales from the spectrum $\operatorname{Max} A'$ of the C*-algebra A' to the spectrum $\operatorname{Max} A$ of the C*-algebra A determined by a map of C*-algebras from A to A' is clearly a map of involutive quantales, in the following sense:

Definition 3.2. By an *involutive map* $\varphi : Q \to Q'$ *of quantales* is meant a map of quantales from an involutive quantale Q to an involutive quantale Q' of which the inverse image homomorphism satisfies the condition that

$$\varphi^*(a^*) = \varphi^*(a)^*$$

for each $a \in Q'$.

Again, it may be noted that any map of locales is necessarily a map of involutive quantales with respect to the canonical involution given by the identity mapping.

The existence of approximate identities in any C*-algebra A allows it to be shown that the involutive quantale $\operatorname{Max} A$ satisfies an important condition which generalises that characterising locales amongst quantales. To describe this property, we recall firstly that an element $a \in Q$ of a quantale Q is said to be *right-sided* provided that

$$a\&1_Q \le a,$$

in which $1_Q \in Q$ denotes the top element of the complete lattice Q. It may be remarked that this inequality is actually an equality in the case that the quantale Q is unital. One similarly defines the element $a \in Q$ to be *left-sided* provided that

$$1_Q \& a \leq a,$$

and *two-sided* provided that it is both left- and right-sided.

Definition 3.3. By a *Gelfand quantale Q* is meant a quantale which is unital, involutive, and for which

$$a = a \& a^* \& a$$

for each right-sided (equivalently, left-sided) element $a \in Q$.

It may be remarked that the equivalence of the definitions in terms of left- and of right-sided elements is due to the fact that the involution* of a Gelfand quantale interchanges right- and left-sided elements, whilst leaving invariant the condition defining the concept of a Gelfand quantale. Indeed, since the property of right- or left-sidedness is also equationally definable, it follows that the concept of a Gelfand quantale is of a particularly straightforward kind.

That the spectrum $\mathrm{Max}\,A$ of a C*-algebra A is a Gelfand quantale follows by an argument which is essentially that of the construction of approximate identities in an arbitrary C*-algebra [8]. Defining the category of Gelfand quantales to be the full subcategory of the category of unital involutive quantales determined by the Gelfand quantales, one therefore has the following:

Theorem 3.1. *The spectrum* $\mathrm{Max}\,A$ *of a* C**-algebra A determines a functor*

$$\mathrm{Max} : (\mathrm{C}^*\texttt{-Algebras})^{op} \to \texttt{Gelfand Quantales}$$

from the dual of the category of C**-algebras to the category of Gelfand quantales.*

It may be remarked that any locale L is canonically a Gelfand quantale with respect to the trivial involution. Conversely, a Gelfand quantale Q is a locale exactly if the element $1_Q \in Q$ is its unit [8].

The relationship between Gelfand quantales and locales is even more intimate than these observations indicate. Denoting by $\mathrm{L}(Q)$, $\mathrm{R}(Q)$ and $\mathrm{I}(Q)$ the subsets of Q consisting respectively of left-, right-, and two-sided elements of Q, a Gelfand quantale Q may be considered to be arranged in the form

$$\begin{tikzcd} & Q & \\ L(Q) \arrow[ur, tail] & & R(Q) \arrow[ul, tail] \\ & I(Q) \arrow[ul, tail] \arrow[ur, tail] & \end{tikzcd}$$

in which the arrows are the inverse image homomorphisms of maps of quantales that are quotient maps in the category of quantales. It may be remarked that these maps are neither unital, nor involutive. Indeed, the involution * provides an anti-isomorphism, with respect to the product, between $\mathrm{R}(Q)$ and $\mathrm{L}(Q)$. The element $1_Q \in Q$ is a right-unit for the quantale $\mathrm{R}(Q)$ and a left-unit for the quantale $\mathrm{L}(Q)$. Moreover, the quantales $\mathrm{R}(Q)$ and $\mathrm{L}(Q)$ are idempotent, in the sense that

$$a\&a = a$$

for any element $a \in \mathrm{R}(Q)$ and any element $a \in \mathrm{L}(Q)$. The element $1_Q \in Q$ is a unit for the quantale $\mathrm{I}(Q)$, and hence, since $\mathrm{I}(Q)$ is also idempotent by the above remarks, it follows that $\mathrm{I}(Q)$ is necessarily a locale, on which, by the Gelfand condition on Q, the involution of Q acts trivially. For, given $a \in \mathrm{I}(Q)$, one has that $a = a\&a^*\&a \leq 1_Q\&a^*\&1_Q \leq a^*$, since necessarily $a^* \in \mathrm{I}(Q)$. Hence, $a \leq a^* \leq a$ on interchanging $a, a^* \in \mathrm{I}(Q)$, yielding that every element of $\mathrm{I}(Q)$ is necessarily self-adjoint.

Since for the spectrum $\mathrm{Max}\,A$ of a C*-algebra A, the quantales $\mathrm{R}(\mathrm{Max}\,A)$ and $\mathrm{L}(\mathrm{Max}\,A)$ are respectively those of closed right and closed left ideals of A, the locale $\mathrm{I}(\mathrm{Max}\,A)$ is therefore that of closed ideals of the C*-algebra A. In the case of a commutative C*-algebra A, this locale $\mathrm{I}(\mathrm{Max}\,A)$ is therefore exactly the classical spectrum of the commutative C*-algebra A, that is to say, the locale of the space of maximal ideals of A. Observing that this assignment is functorial on the dual of the category of commutative C*-algebras yields the following:

Corollary 3.2. *The functor*

$$\mathrm{I}(\mathrm{Max})\colon (\mathtt{Commutative}\ C^*\text{-}\mathtt{Algebras})^{op} \to \mathtt{Locales}$$

which assigns to each commutative C*-*algebra* A *the locale* $\mathrm{I}(\mathrm{Max}\,A)$ *is exactly the classical spectrum functor on the dual of the category of commutative* C*-*algebras.*

It will later have logical significance that this construction may also be obtained by observing that any Gelfand quantale Q admits a coreflection into the category of locales, which in the case of a Gelfand quantale which is commutative is given by taking the locale $\mathrm{I}(Q)$ of two-sided elements of Q, of which the coadjunction is the embedding

$$\iota_Q\colon \mathrm{I}(Q) \to Q$$

of Gelfand quantales, of which the inverse image homomorphism assigns to each element $q \in Q$ of the quantale its two-sided closure $1_Q\&q\&1_Q \in \mathrm{I}(Q)$. In particular, the classical spectrum of a commutative C*-algebra A is therefore exactly the coreflection into the category of locales of its spectrum $\mathrm{Max}\,A$ in the present sense.

4. The theory $\mathbb{M}\mathrm{ax}\,A$

In the case of a commutative C*-algebra A, the classical spectrum $\mathrm{Max}\,A$, considered as a locale, may be obtained constructively by introducing [2,5] a propositional geometric theory $\mathbb{M}\mathrm{ax}\,A$. The spectrum $\mathrm{Max}\,A$ in the classical sense is then constructed by taking the Lindenbaum algebra of this theory within constructive logic, in a sense to be made more precise below. To avoid confusion with the concept of spectrum for an arbitrary C*-algebra A, as considered in the preceding section, the classical spectrum of a commutative C*-algebra A will be denoted by

$$\mathrm{Max}_{Loc}\, A,$$

to indicate that the spectrum of the C*-algebra is being constructed as a locale. Of course, the spectrum in this classical sense is actually the topological space obtained by observing that the locale $\mathrm{Max}_{Loc}\, A$ is indeed spatial. In particular, the points are then the models of the propositional geometric theory $\mathbb{M}\mathrm{ax}\, A$.

The theory which canonically generates the locale $\mathrm{Max}_{Loc}\, A$ in this case is that which introduces a proposition

$$a \in P$$

for each element $a \in A$ of the commutative C*-algebra A, together with axioms given by:

$$\begin{array}{lll} & true & \vdash\ 1_A \in P \\ & 0_A \in P & \vdash\ false \\ & a + b \in P & \vdash\ a \in P \vee b \in P \\ & ab \in P & \dashv\vdash a \in P \wedge b \in P \\ \text{and} & a \in P & \vdash\ \bigvee_i a_i \in P \qquad \text{whenever } a \in \overline{\sum_i a_i} \end{array}$$

for each $a, b \in A$ and $a_i \in A$. The notation $\overline{\sum_i a_i}$ is used here to denote the closed linear subspace of A generated by the elements $a_i \in A$.

This description may be applied to construct the locale $\mathrm{Max}_{Loc}\, A$ in either of two ways: on the one hand, the theory considered is that of (the complement P of) a closed prime ideal of the commutative C*-algebra A, the closed prime ideals being exactly the maximal ideals of A. The locale $\mathrm{Max}_{Loc}\, A$ is then that of propositions in this theory, ordered by provable entailment in the theory, modulo provable equivalence in the theory. The points of the locale, which are logically the classical models of the theory, in the sense of validations of its axioms in the Boolean algebra **2**, are therefore the maximal ideals of the commutative C*-algebra A. In particular, the locale is that of the classical spectrum of the C*-algebra.

On the other hand, the locale $\mathrm{Max}_{Loc}\, A$ may equally be considered lattice-theoretically to be that generated by the symbols $a \in P$ introduced for each $a \in A$, subject to the relations expressed by the axioms of the theory, with entailment $\vdash$ being interpreted by the order relation $\leq$. In this case there is an algebraic description of the locale $\mathrm{Max}_{Loc}\, A$ deriving from these generators and relations provided by the theory. Once again, the points of the locale correspond to interpretations of its generators and validations of its relations in the Boolean algebra **2**, hence to completely prime filters in the locale. The advantage of this algebraic description is its constructional simplicity, while that of the logical approach is motivational, in that one just writes down constructively the theory that classically describes the points of the spectrum.

In considering a generalisation of this construction to the case of an arbitrary C*-algebra A, one finds oneself disadvantaged in a number of ways. Whilst the logical framework that will produce an involutive quantale as the Lindenbaum algebra of a propositional theory is fairly straightforward to describe, the nature of the theory to be considered is less evident. In part, this is because the concept of a classical model of such a theory, in the sense of a quantised version of a validation in the Boolean algebra **2**, is at this stage undefined. Further, although it might be anticipated that, whatever the concept of such a model should be, it would be likely to relate in this case to the irreducible representations of the C*-algebra A, the exact way in which this correspondence should emerge is not at the outset apparent. Indeed, examining the spectrum of a C*-algebra from this point of view might be expected to throw some light on these more general questions.

The approach which may be taken is to place emphasis on what might be expected to be another aspect of the construction. In the case of a commutative C*-algebra A, it might be hoped that the spectrum $\mathrm{Max}\,A$ should have the property that its reflection into the category of locales should coincide with the classical spectrum, $\mathrm{Max}_{Loc}\,A$, of the commutative C*-algebra A. In the next section, it will be shown directly that this is indeed the case for the spectrum $\mathrm{Max}\,A$ introduced in the preceding section. However, this may also be viewed from a logical standpoint, in which the theory $\mathbb{M}\mathrm{ax}\,A$ to be introduced should simply be such that, when interpreted within the commutative context of ordinary constructive logic, rather than that of the non-commutative yet involutive world which leads to quantales of the kind of $\mathrm{Max}\,A$, it should yield the classical spectrum $\mathrm{Max}_{Loc}\,A$ of the commutative C*-algebra A.

Taking this approach in the most straightforward manner possible, by simply rewriting the theory described above, only now within a non-commutative logical context, leads to consideration of the following theory:

Definition 4.1. By the *theory of the spectrum* of a C*-algebra A will be meant the theory $\mathbb{M}\mathrm{ax}\,A$ obtained by introducing for each element $a \in A$ of the C*-algebra A a proposition

$$a \in P,$$

together with the following axioms:

$$\begin{array}{lll}
\text{(P1)}\ \mathit{true} & \vdash & 1_A \in P \\
\text{(P2)}\ 0_A \in P & \vdash & \mathit{false} \\
\text{(P3)}\ a^* \in P & \vdash & a \in P^* \\
\text{(P4)}\ a + b \in P & \vdash & a \in P \vee b \in P \\
\text{(P5)}\ ab \in P & \dashv\vdash & a \in P \,\&\, b \in P \\
\text{(P6)}\ a \in P & \vdash & \bigvee_i a_i \in P \qquad \text{whenever } a \in \overline{\sum_i a_i}.
\end{array}$$

It will be observed that the only difference from the theory considered in the commutative case is the appearance of the non-commutative conjunction & in the axiom (P5), and the presence of the axiom (P3) relating the involution of the C*-algebra to that of the propositional language. For the moment, the presence of this involution within the propositional language will be motivated only by stating that intuitively, to the extent that the conjunction & is intended to be interpreted as meaning *and then*, the involution is intended to represent reversal of the implied arrow of time.

More formally, the deductive system within which the theory is to be considered is therefore that of which the semantics are given by unital involutive quantales, hence the Lindenbaum algebra of the theory $\mathbb{M}\mathrm{ax}\,A$, consisting of propositions of the theory, ordered by provable entailment in the theory, modulo provable equivalence in the theory, is a unital involutive quantale. Concerning this quantale, which will be referred to simply as the *quantale of the theory* $\mathbb{M}\mathrm{ax}\,A$, and which may equally be obtained lattice theoretically by considering the propositions of the theory as generators, and the axioms of the theory as relations, one has the following:

Theorem 4.1. *The quantale of the theory* $\mathbb{M}\mathrm{ax}\,A$ *of the spectrum of a* C*-*algebra* A *is canonically isomorphic to the spectrum*

$$\mathrm{Max}\,A$$

of the C*-*algebra* A, *by the homomorphism of unital involutive quantales determined by the assignment to each primitive proposition*

$$a \in P$$

of the theory of the closed linear subspace generated by the element $a \in A$ *of the* C*-*algebra* A.

Proof. Consider then the assignment to each primitive proposition $a \in P$ of the, necessarily closed, linear subspace $\langle a \rangle \in \mathrm{Max}\,A$ of the C*-algebra A. Observe that each of the axioms is validated in the quantale $\mathrm{Max}\,A$, since $0_A \in P$ maps to $\langle 0_A \rangle$, which is indeed the interpretation of *false*, while $1_A \in P$ maps to $\langle 1_A \rangle$, which is the interpretation of *true*. Further, $a^* \in P$ maps to the closed linear subspace $\langle a^* \rangle$ generated by the involute of $a \in A$, which is contained in, and in fact equal to, the involute of the closed linear subspace $\langle a \rangle$ generated by $a \in A$. Similarly, the element $a + b \in A$ belongs to the subspace generated by the elements $a, b \in A$, hence $\langle a + b \rangle \leq \langle a \rangle \vee \langle b \rangle$, and $\langle ab \rangle$ is exactly $\langle a \rangle \& \langle b \rangle$, by the definition of the product of the quantale $\mathrm{Max}\,A$. Finally, if $a \in \overline{\sum_i a_i}$, then $\langle a \rangle \leq \overline{\sum_i a_i} = \bigvee_i \langle a_i \rangle$. The assignment therefore determines a homomorphism of unital involutive quantales from the quantale of the theory $\mathbb{M}\mathrm{ax}\,A$ to the quantale $\mathrm{Max}\,A$.

Now we assert that this is an isomorphism of quantales. To achieve this, it suffices to show that the assignment described above is universal, in the sense that any mapping $\varphi : A \to Q$ from the set of primitive propositions (which will be tacitly identified with the set A of elements of the C*-algebra A) to a unital involutive quantale Q

which validates the axioms of the theory factors uniquely through a homomorphism $\Phi\colon \operatorname{Max} A \to Q$ of unital involutive quantales. To see this, suppose that such a mapping $\varphi : A \to Q$ is given, and define a mapping $\Phi\colon \operatorname{Max} A \to Q$ by setting

$$\Phi(M) = \bigvee_{a \in M} \varphi(a)$$

for each $M \in \operatorname{Max} A$, and assert that

$$\Phi(M \& N) = \Phi(M) \& \Phi(N),$$

$$\Phi\left(\bigvee_i M_i\right) = \bigvee_i \Phi(M_i)$$

$$e_Q \le \Phi(e_{\operatorname{Max} A})$$

for any $M, N \in \operatorname{Max} A$ and any family $M_i \in \operatorname{Max} A$.

Firstly, consider the product $M \& N$ of closed linear subspaces $M, N \in \operatorname{Max} A$. By definition of the mapping Φ, one has that

$$\Phi(M \& N) = \bigvee_{c \in M \& N} \varphi(c).$$

For any $c \in M \& N$, one has, by definition of the product $M \& N$ in the quantale $\operatorname{Max} A$, that $c \in \overline{\sum_{a \in M, b \in N} \langle ab \rangle}$, hence that

$$c \in P \vdash \bigvee_{a \in M, b \in N} ab \in P$$

by the axiom (P6) of the theory $\mathbb{M}\mathrm{ax}\, A$. Hence,

$$\varphi(c) \le \bigvee_{a \in M, b \in N} \varphi(ab),$$

since the mapping φ validates the axioms of $\mathbb{M}\mathrm{ax}\, A$.

Again, by the axiom (P5) of $\mathbb{M}\mathrm{ax}\, A$, one has that

$$ab \in P \dashv\vdash a \in P \& b \in P$$

for each $a \in M$, $b \in N$, and hence that

$$\varphi(ab) = \varphi(a) \& \varphi(b)$$

in the quantale Q. Thus,

$$\begin{aligned}\Phi(M \& N) &= \bigvee_{c \in M \& N} \varphi(c) \\ &\le \bigvee_{a \in M, b \in N} \varphi(ab) \\ &= \bigvee_{a \in M, b \in N} \varphi(a) \& \varphi(b)\end{aligned}$$

$$= \bigvee_{a\in M} \varphi(a) \& \bigvee_{b\in N} \varphi(b)$$

$$= \Phi(M) \& \Phi(N).$$

The converse is obtained by observing trivially that, in fact,

$$\bigvee_{c\in M\&N} \varphi(c) = \bigvee_{a\in M, b\in N} \varphi(ab),$$

by the observation that $a \in M$ and $b \in N$ implies that $ab \in M\&N$. Hence,

$$\Phi(M\&N) = \Phi(M)\&\Phi(N)$$

for each $M, N \in \operatorname{Max} A$. The condition that

$$\Phi\left(\bigvee_i M_i\right) = \bigvee_i \Phi(M_i)$$

is proved by observing that

$$\Phi\left(\bigvee_i M_i\right) = \bigvee_{c\in\bigvee_i M_i} \varphi(c) \leq \bigvee_i \bigvee_{a_i\in M_i} \varphi(a_i) = \bigvee_i \Phi(M_i),$$

by an application of (P6) to the fact that $c \in \bigvee_i M_i$ implies $c \in \overline{\sum_i \sum_{a_i\in M_i} \langle a_i \rangle}$, whilst the converse follows trivially since $a_j \in M_j$ for any j implies that $a_j \in \bigvee_i M_i$. In particular, this shows that

$$\Phi(0_{\operatorname{Max} A}) = 0_Q.$$

Finally, we have that

$$e_Q \leq \Phi(e_{\operatorname{Max} A})$$

since $e_{\operatorname{Max} A}$ is the closed linear subspace $\langle 1_A \rangle \in \operatorname{Max} A$, hence

$$\Phi(e_{\operatorname{Max} A}) = \bigvee_{a\in\langle 1_A\rangle} \varphi(a) \geq \varphi(1_A) \geq e_Q,$$

since the mapping φ validates the axiom (P1) of the theory $\mathbb{M}\mathrm{ax}\, A$. $\square$

It may be remarked that the proof would also apply to showing that the spectrum $\operatorname{Max} A$ is obtained by omitting from the theory of the spectrum the involutive axiom (P3) and carrying out the interpretation in the category of unital quantales, rather than of unital involutive quantales. The quantale of the theory may in that case be shown to have a canonical involution, induced by defining the involute of the primitive proposition

$$a \in P$$

to be the proposition

$$a^* \in P$$

determined by the involute of $a \in A$, and with respect to this involution the canonical isomorphism with the spectrum $\mathrm{Max}\,A$ is indeed an isomorphism of unital involutive quantales.

The translation of logical constructs involving the theory $\mathbb{M}\mathrm{ax}\,A$ to algebraic realisations in terms of the quantale $\mathrm{Max}\,A$ is completed by the following observations:

Definition 4.2. By a *model* of the theory $\mathbb{M}\mathrm{ax}\,A$ in a unital involutive quantale Q will be meant an assignment to each primitive proposition

$$a \in P$$

of the theory of an element $[\![a \in P]\!]$ of the quantale Q, in such a way that the axioms of the theory are validated in the quantale Q.

The validation required in the above definition is that derived by interpreting entailment by the order relation of the quantale, and the logical connectives of the theory by the corresponding operations of the quantale. In particular, the logical constants *true* and *false* are to be interpreted respectively by the unit $e_Q \in Q$ and the zero $0_Q \in Q$ of the quantale Q. As a consequence, whilst an axiom requiring that a proposition entails *false* is validated exactly if the proposition is interpreted by the zero element $0_Q \in Q$, an assertion that a proposition is entailed by *true* is validated whenever the proposition is assigned a truth value lying *above* the unit element $e_Q \in Q$.

The description of the quantale of the theory in terms of generators and relations yields immediately the following:

Corollary 4.2. *The models of the theory* $\mathbb{M}\mathrm{ax}\,A$ *of the spectrum of the* C^**-algebra* A *in any unital involutive quantale* Q *correspond exactly to the homomorphisms*

$$\mathrm{Max}\,A \to Q$$

of unital involutive quantales from the spectrum $\mathrm{Max}\,A$ *of the* C^**-algebra* A *to the quantale* Q.

It may be remarked that it is to obtain this equivalence that the definition of a map

$$\varphi : Q \to Q'$$

of unital involutive quantales has been taken to require only that

$$e_Q \leq \varphi^*(e_{Q'}),$$

rather than equality, corresponding logically to this consideration that a proposition is validated in an interpretation provided that it is assigned a value lying above the unit element of the quantale concerned.

It may be observed that this identification of the homomorphisms

$$\mathrm{Max}\,A \to Q$$

of unital involutive quantales with the models of the theory $\mathbb{M}\mathrm{ax}\,A$ in a quantale Q already indicates that any consideration of the quantale to provide a concept of classical model of the theory, hence a concept of point for the spectrum $\operatorname{Max} A$, is likely to prove inadequate. Indeed, it may be verified immediately that any primitive ideal $\mathfrak{p}$ of the C*-algebra A yields a homomorphism

$$\operatorname{Max} A \to \mathbf{2}$$

of unital involutive quantales by interpreting the proposition

$$a \in P$$

of the theory by *false* in the event that $a \in \mathfrak{p}$ and by *true* otherwise. The significance of this observation in the context of the spectrum $\operatorname{Max} A$ will become apparent at a later stage, once the more appropriate concept of a classical model of the theory has been identified. However, it may be noted in passing that the topological space of primitive ideals of a C*-algebra A has long been known to provide a notion of spectrum that is neither interesting topologically, nor adequate functionally, in terms of the kind of representations that it can sustain [3].

It is to examining the question of finding a more adequate concept of the points of the spectrum, which is likely to be more closely related to that of the irreducible representations of the C*-algebra, that we now turn. In doing so, it may be noted that already, in the analysis of the spectrum considered above, it has proved convenient to work with the category of quantales and inverse image homomorphisms, rather than with its dual, the category of quantales and maps of quantales. This is a convention that will be adopted throughout the rest of the paper, until its final conclusions revert to matters properly considered within the category of quantales and maps of quantales. On occasion, we shall refer to the inverse image homomorphism

$$\varphi^* : Q' \to Q$$

of a map $\varphi : Q \to Q'$ of quantales simply by the symbol φ when from the context it is evident that it is the homomorphism, rather than the map, that is being considered. We shall also refer to the inverse image homomorphism of a map of quantales as a homomorphism of quantales.

5. Representations of C*-algebras

For a commutative C*-algebra A, the spectrum is obtained classically by taking the set of multiplicative linear functionals

$$\varphi : A \to \mathbb{C}$$

together with the weak* topology induced by evaluation of linear functionals. The kernel of a multiplicative linear functional φ is a maximal ideal $\mathfrak{m}_\varphi$ of the C*-algebra A. Moreover, any maximal ideal $\mathfrak{m}$ of A is the kernel of a unique multiplicative linear functional

$$\varphi_{\mathfrak{m}} : A \to \mathbb{C},$$

by the Gelfand–Mazur theorem. The points of the spectrum of A may thus be identified with the set of maximal ideals of the C*-algebra A. Moreover, the weak* topology on the spectrum may be seen to correspond to the hull-kernel, or Zariski, topology on the set of maximal ideals of A.

To see the way in which these ideas develop in passing from a commutative C*-algebra to a C*-algebra in general, we recall the following concepts:

Definition 5.1. By a *representation* of a C*-algebra A on a Hilbert space H is meant a homomorphism

$$\varphi : A \to \mathscr{B}(H)$$

from the C*-algebra A to the C*-algebra $\mathscr{B}(H)$ of bounded linear operators on H.

It may be remarked that, in the particular case when the Hilbert space H is the space $\mathbb{C}$ of complex numbers, then a representation of the C*-algebra A on H is exactly a multiplicative linear functional on A. In the case of a commutative C*-algebra A, it may be shown that any representation may be expressed as a product, in an appropriate sense, of representations on the Hilbert space $\mathbb{C}$ of complex numbers. Indeed, this property provides a characterisation of commutative C*-algebras amongst C*-algebras in general [4].

The introduction of the concept of a representation involves an important change of emphasis in considering non-commutative C*-algebras. We are concerned not only with mapping the C*-algebra A into a C*-algebra $\mathscr{B}(H)$ of bounded linear operators on a Hilbert space H, but also with allowing the C*-algebra A to act geometrically on the elements of H. Explicitly, any representation

$$\varphi : A \to \mathscr{B}(H)$$

of the C*-algebra A on a Hilbert space H allows each element $a \in A$ to act on the Hilbert space H by the bounded linear operator

$$\varphi_a : H \to H,$$

assigning to each $x \in H$ the element $x\,\varphi_a \in H$.

In particular, to each closed linear subspace M of the Hilbert space H there may be assigned by this action of an element $a \in A$ the closed linear subspace

$$M\,\varphi_a = \overline{\{x\varphi_a \in H \mid x \in M\}}$$

of H obtained by taking the closure of the direct image of M under the linear operator $\varphi_a : H \to H$. It may be verified straightforwardly that in this way one obtains for each $a \in A$ a sup-preserving mapping, which by extension will also be denoted

$$\varphi_a : \mathscr{P}(H) \to \mathscr{P}(H),$$

from the sup-lattice $\mathscr{P}(H)$ of closed linear subspaces of H to itself.

As will have been apparent, it will be our convention to write the action of these mappings to the right of their argument. With this in mind, we make [10] the following:

Definition 5.2. By the *quantale $\mathscr{Q}(H)$ of the Hilbert space H* will be meant the quantale of sup-preserving mappings

$$\alpha : \mathscr{P}(H) \to \mathscr{P}(H)$$

from the sup-lattice $\mathscr{P}(H)$ of closed linear subspaces of H to itself.

The operations with respect to which $\mathscr{Q}(H)$ is a quantale are with respect to the pointwise ordering on sup-preserving mappings on $\mathscr{P}(H)$, together with the product given by composition. Explicitly, the supremum on $\mathscr{Q}(H)$ is given by

$$M\left(\bigvee_i \alpha_i\right) = \bigvee_i M\alpha_i.$$

The quantale $\mathscr{Q}(H)$ is also unital, with unit given by the identity mapping on $\mathscr{P}(H)$, and involutive, with respect to the involution given by

$$M\,\alpha^* = \left(\bigvee_{N\alpha \leq M^\perp} N\right)^\perp,$$

in which $^\perp$ denotes taking the *orthogonal complement* of a closed linear subspace of H. It is known further [10] that with respect to these operations the quantale

$$\mathscr{Q}(H)$$

of a Hilbert space H is a Gelfand quantale.

Theorem 5.1. *Any representation of a* C*-*algebra A on a Hilbert space H determines a homomorphism of Gelfand quantales*

$$\mathrm{Max}\,A \to \mathscr{Q}(H)$$

from the spectrum of the C*-*algebra A to the quantale $\mathscr{Q}(H)$ of the Hilbert space H.*

Proof. By the results of the preceding section, it is enough to show that a representation

$$\varphi : A \to \mathscr{B}(H)$$

of the C*-algebra A on a Hilbert space H determines a model of the theory $\mathbb{M}\mathrm{ax}\,A$ in the quantale $\mathscr{Q}(H)$ of the Hilbert space. So, consider the interpretation which assigns to each primitive proposition of the theory $\mathbb{M}\mathrm{ax}\,A$ the element

$$\varphi_a : \mathscr{P}(H) \to \mathscr{P}(H)$$

defined above of the quantale $\mathscr{Q}(H)$. Then the axioms of the theory $\mathbb{M}\mathrm{ax}\,A$ are validated in $\mathscr{Q}(H)$ by the following arguments: firstly, the axioms $true \vdash 1 \in P$ and $0 \in P \vdash false$ interpret respectively as $\iota_H \leq \varphi_{1_A}$ and $\varphi_{0_A} \leq o_H$ in which $\iota_H, o_H \in \mathscr{Q}(H)$ are respectively the identity mapping and the zero mapping on the sup-lattice $\mathscr{P}(H)$. But, the representation $\varphi : A \to \mathscr{B}(H)$ maps the identity element $1_A \in A$ and the zero element $0_A \in A$ respectively to the identity operator and the zero operator on the Hilbert space H, which implies that these axioms are indeed validated.

Now consider the axiom (P3), which requires that $a^* \in P \vdash a \in P^*$ for any $a \in A$. Recalling that the right-hand side of this represents the involute of the primitive proposition $a \in P$, it must be shown that $\varphi_{a^*} \leq \varphi_a^*$, in which the right-hand side represents the involute of the element $\varphi_a \in \mathscr{Q}(H)$ of the quantale of the Hilbert space H. To show that this condition is satisfied, it is enough to show that for any closed linear subspace M of the Hilbert space, we have that $M\varphi_{a^*} \leq M\varphi_a^*$. Now, the action of the involute of $\varphi_a \in \mathscr{Q}(H)$ on the element $M \in \mathscr{P}(H)$ is given by the expression

$$M\varphi_a^* = \left(\bigvee_{N\varphi_a \leq M^\perp} N\right)^\perp,$$

in which $^\perp$ denotes orthogonal complementation in the sup-lattice of closed linear subspaces of H. To verify, then, that $M\varphi_{a^*}$ is contained in the orthogonal complement of the largest linear subspace of which the closure of the image under the linear operator φ_a on H is contained in the orthogonal complement of M, it suffices to show that for any $x \in M$, for any closed linear subspace N of which the image $N\varphi_a$ is contained in the orthogonal complement $M^\perp$ of the closed linear subspace M, and for any $y \in N$, one has that

$$\langle x\varphi_{a^*}, y\rangle = 0$$

in the inner product of the Hilbert space H. But, since the representation $\varphi : A \to \mathscr{B}(H)$ is an involutive homomorphism, the linear operator $\varphi_{a^*} : H \to H$ assigned to the involute $a^* \in A$ of the element $a \in A$ is exactly the adjoint $\varphi_a^* : H \to H$ of the linear operator $\varphi_a : H \to H$. Hence,

$$\langle x\varphi_{a^*}, y\rangle = \langle x\varphi_a^*, y\rangle = \langle x, y\varphi_a\rangle.$$

But, $y \in N$ implies that $y\varphi_a \in N\varphi_a$ is orthogonal to any $x \in M$, since $N\varphi_a \leq M^\perp$. The inner product $\langle x, y\varphi_a\rangle$ is therefore zero, as required, validating the axiom (P3).

The axioms $a + b \in P \vdash a \in P \vee b \in P$ and $ab \in P \vdash\dashv a \in P \& b \in P$ are validated respectively because $x\varphi_{a+b} = x\varphi_a + x\varphi_b \in M\varphi_a \vee M\varphi_b$ and $x\varphi_{ab} = (x\varphi_a)\varphi_b$ for any closed linear subspace M and any $x \in M$, by the additivity and the multiplicativity of the representation, thereby verifying (P4) and (P5). And finally, concerning the axiom (P6), $a \in P \vdash \bigvee_i a_i \in P$ whenever $a \in A$ lies in the closed linear subspace generated by the elements $a_i \in A$ is validated provided that

$$M\varphi_a \leq \bigvee_i M\varphi_{a_i}$$

for each closed linear subspace M of the Hilbert space. However, choosing a sequence of elements $b_n \in A$ each lying in the linear subspace generated by the elements $a_i \in A$ such that $b_n \to a$ in the C*-algebra A, we have that $\varphi_{b_n} \to \varphi_a$ in the C*-algebra $\mathscr{B}(H)$, by the continuity of the representation. Hence, for any $x \in M$ we have that $x\varphi_{b_n} \to x\varphi_a$ in the Hilbert space H, from which it follows that $x\varphi_a \in M\varphi_a$ lies in the closure of the linear subspace of H generated by the elements $x\varphi_{b_n} \in H$. But, since each $b_n \in A$ may be expressed as a linear combination of the elements $a_i \in A$, it follows that $x\varphi_{b_n} \in \bigvee_i M\varphi_{a_i}$ as required.

Hence, the interpretation indeed validates the axioms of the theory $\mathbb{M}\mathrm{ax}\, A$, so determines a homomorphism of unital involutive quantales from the spectrum $\mathrm{Max}\, A$ of the C*-algebra A to the quantale $\mathscr{Q}(H)$ of the Hilbert space H. □

Concerning representations of a C*-algebra, we recall [4] the following:

Definition 5.3. By an *equivalence* of representations $\varphi : A \to \mathscr{B}(H)$, $\varphi' : A \to \mathscr{B}(H')$ of a C*-algebra A on Hilbert spaces H, H' is meant an isomorphism

$$\eta : H \to H'$$

of Hilbert spaces for which the corresponding isomorphism $\sigma_\eta : \mathscr{B}(H) \to \mathscr{B}(H')$ makes the diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[dr, "\varphi'"'] & \mathscr{B}(H) \arrow[d, "\sigma_\eta"] \\ & \mathscr{B}(H') \end{tikzcd}$$

commute.

It may be proved straightforwardly that given such equivalent representations of a C*-algebra A, the homomorphisms $\mathrm{Max}\, A \to \mathscr{Q}(H)$ and $\mathrm{Max}\, A \to \mathscr{Q}(H')$ thereby determined from the spectrum of the C*-algebra A into the quantales of the Hilbert spaces of the representations are such that there exists an isomorphism $\mathscr{P}(H) \to \mathscr{P}(H')$ of orthocomplemented sup-lattices induced by the isomomorphism $\eta : H \to H'$ of Hilbert spaces, for which the corresponding isomorphism $\mathscr{Q}(H) \to \mathscr{Q}(H')$ makes the diagram

$$\begin{tikzcd} \mathrm{Max}\, A \arrow[r] \arrow[dr] & \mathscr{Q}(H) \arrow[d] \\ & \mathscr{Q}(H') \end{tikzcd}$$

commute. In other words, equivalent representations of C*-algebras yield homomorphisms of Gelfand quantales that are equivalent in the evident sense.

Now, recall that a closed linear subspace M of the Hilbert space H is said to be *invariant* under the representation provided that

$$x \in M \quad \text{implies } x\varphi_a \in M$$

for all $a \in A$. One then has the following:

Definition 5.4. A representation

$$\varphi : A \to \mathscr{B}(H)$$

is said to be *irreducible* provided that it is not zero, and that the only closed linear subspaces of the Hilbert space H that are invariant under the representation are the zero subspace and the Hilbert space H itself.

The concept of an irreducible representation has importance within the theory of representations of C*-algebras because it may be shown [4] that any representation of a C*-algebra A is equivalent to a direct sum of irreducible representations of A. The existence of irreducible representations for any C*-algebra A yields the Gelfand–Naimark representation of A as a closed involutive subalgebra of the C*-algebra

$$\mathscr{B}(H)$$

of bounded linear operators on a Hilbert space H constructed from the Hilbert spaces on which the C*-algebra A admits irreducible representations.

More particularly, in the context of the present discussion, the irreducible representations of a commutative C*-algebra A are each on the Hilbert space $\mathbb{C}$ of complex numbers, hence correspond to the multiplicative linear functionals on the C*-algebra A. The classical spectrum of a commutative C*-algebra A is therefore exactly the space of irreducible representations of A. The multiplicative linear functionals, moreover, correspond exactly to the maximal ideals of the commutative C*-algebra A, of which each is the kernel of a unique multiplicative linear functional. In the case of a C*-algebra A which is not necessarily commutative, this correspondence between irreducible representations of A and, in this case, maximal right ideals of A persists, although the relationship, which we shall now recall, is rather more subtle than in the case of a commutative C*-algebra.

Given an irreducible representation

$$\varphi : A \to \mathscr{B}(H)$$

of the C*-algebra A on a Hilbert space H, for any non-zero element $x \in H$ the subset

$$x\varphi_A = \overline{\{x\varphi_a \in H \mid a \in A\}}$$

is a closed linear subspace of H which is invariant under the representation. Since it is necessarily non-zero, it is therefore equal to the Hilbert space H itself. Consider now the subset

$$\mathfrak{m}_x = \{a \in A \mid x\varphi_a = 0_H\}$$

of the C*-algebra A. Then, $\mathfrak{m}_x$ is evidently a right ideal of the C*-algebra A, closed since it is the inverse image of $0_H \in H$ under the bounded linear mapping

$$\mu_x : A \to H$$

which assigns to each $a \in A$ the image under $\varphi_a \in \mathscr{B}(H)$ of the element $x \in H$. Moreover, the closed right ideal $\mathfrak{m}_x$ is a maximal right ideal of the C*-algebra A, of which the quotient space $A/\mathfrak{m}_x$ is canonically isomorphic to the Hilbert space H by the linear mapping induced by $\mu_x : A \to H$. The representation

$$\varphi_x : A \to \mathscr{B}(A/\mathfrak{m}_x)$$

thereby determined is canonically equivalent to the irreducible representation

$$\varphi : A \to \mathscr{B}(H)$$

which determines it, by the canonical isomorphism

$$\eta_x : A/\mathfrak{m}_x \to H$$

of Hilbert spaces. Conversely, for any maximal right ideal $\mathfrak{m}$ of the C*-algebra A, the quotient space $A/\mathfrak{m}$ is canonically a Hilbert space with respect to the inner product determined by the pure state

$$\pi_\mathfrak{m} : A \to \mathbb{C}$$

of the C*-algebra A corresponding to the maximal right ideal $\mathfrak{m}$. Explicitly, the maximality of the closed right ideal $\mathfrak{m}$ is equivalent to that of the self-adjoint, closed linear subspace $\mathfrak{n}$ obtained by setting

$$\mathfrak{n} = \mathfrak{m} + \mathfrak{m}^*,$$

which is therefore the kernel of a linear functional

$$\pi_\mathfrak{m} : A \to \mathbb{C}$$

yielding the pure state. In turn, the inner product on the quotient space $A/\mathfrak{m}$ is then that defined by setting

$$\langle a + \mathfrak{m}, b + \mathfrak{m} \rangle = \pi_\mathfrak{m}(ab^*)$$

for each $a, b \in A$. Evidently, the C*-algebra A admits a canonical action on the quotient space $A/\mathfrak{m}$ by right multiplication, yielding a representation

$$\varphi_\mathfrak{m} : A \to \mathscr{B}(A/\mathfrak{m})$$

of the C*-algebra A on the Hilbert space $A/\mathfrak{m}$, which is irreducible by the maximality of the closed right ideal $\mathfrak{m}$.

It may be remarked in passing that the maximal right ideal $\mathfrak{m}$ may be recovered from the kernel $\mathfrak{n}$ of the pure state $\pi_\mathfrak{m}$ which it determines, as the largest closed right ideal $\mathfrak{m}$ contained in $\mathfrak{n}$. It may also be verified straightforwardly that the maximal right ideal of the C*-algebra A obtained by applying the above construction to the element

$$1_A + \mathfrak{m} \in A/\mathfrak{m}$$

corresponding to the identity element $1_A \in A$ is exactly the maximal right ideal $\mathfrak{m}$ of A from which the irreducible representation has been obtained. In this way, one obtains the canonical correspondence between the maximal right ideals $\mathfrak{m}$, the pure states $\pi_\mathfrak{m}$, and the canonical irreducible representations $\varphi_\mathfrak{m}$ of the C*-algebra A, to which we later make reference.

For the moment, we note from these observations the following:

Corollary 5.2. *Any irreducible representation of the* C**-algebra A on a Hilbert space H determines a homomorphism from the spectrum* Max$\,A$ *to the quantale of the Hilbert space H which is equivalent to the homomorphism*

$$\varphi_{\mathfrak{m}} : \operatorname{Max} A \to \mathscr{Q}(A/\mathfrak{m})$$

determined by a maximal right ideal $\mathfrak{m}$, *for some maximal right ideal* $\mathfrak{m}$ *of the* C**-algebra A.*

Finally, recalling that an ideal $\mathfrak{p}$ of a C*-algebra A is said to be *primitive* provided that it is the largest ideal contained in some maximal right ideal $\mathfrak{m}$ of the C*-algebra A, it may be noted further, in the light of the construction considered above, that the primitive ideal $\mathfrak{p}$ which is the kernel of the representation is exactly that determined by the expression:

$$\mathfrak{p} = \{a \in A \mid a \in P \models \mathit{false}\},$$

in which $\models$ denotes entailment in the model of the theory $\mathbb{M}$ax A determined by the representation. In turn, the primitive ideal $\mathfrak{p}$ is equally given by the intersection:

$$\mathfrak{p} = \bigcap_{x \in H,\, x \neq 0} \mathfrak{m}_x$$

of all the maximal right ideals associated with the irreducible representation. At a later point, it will be possible to interpret this observation logically in terms of a reflection of the model obtained from a maximal right ideal $\mathfrak{m}$ to a model within constructive, rather than physical, logic.

6. Hilbert quantales

The concept of a point of a Gelfand quantale, such as the spectrum Max$\,A$ of a C*-algebra A, should be independent of the particular kind of quantale considered. The indications flowing from the preceding discussion are that, in the case of the quantale Max$\,A$, the points should turn out to be the homomorphisms

$$\varphi : \operatorname{Max} A \to \mathscr{Q}(H)$$

induced by the irreducible representations of the C*-algebra A. Moreover, in that case, the points may be considered to correspond to the maximal right ideals of the C*-algebra A. It must now be seen how the quantales $\mathscr{Q}(H)$ arise intrinsically from the C*-algebra A, by first abstracting to certain quantales which may be considered to have a natural role to play in the context of classical models of theories.

There are two strands which have emerged in the discussion above. One is that in moving from the commutative case, one has had to replace multiplicative linear functionals into the complex numbers by representations on Hilbert space. The other is that an important role is played by the sup-lattice $\mathscr{P}(H)$ of closed linear subspaces of the Hilbert space H. In this latter context, the involutive aspect of the representation has been linked inextricably with the operation of orthocomplementation which is present in the sup-lattice $\mathscr{P}(H)$.

These ideas, of allowing a Gelfand quantale Q to act on an orthocomplemented sup-lattice S by means of a homomorphism

$$\varphi : Q \to \mathscr{Q}(S)$$

of Gelfand quantales from the quantale Q to the quantale $\mathscr{Q}(S)$ of sup-preserving homomorphisms on an orthocomplemented sup-lattice S, are not unfamiliar [10] in the context of Gelfand quantales, of which we begin by recalling some aspects.

Definition 6.1. By a *Hilbert quantale* Q is meant any quantale which is isomorphic to the quantale

$$\mathscr{Q}(S)$$

of sup-preserving mappings from an orthocomplemented sup-lattice S to itself. The Hilbert quantale $\mathscr{Q}(S)$ will itself be referred to as the *quantale of the orthocomplemented sup-lattice* S.

It is recalled that by an *orthocomplemented sup-lattice* S is meant a sup-lattice S together with an operation $^{\perp}$ of orthocomplementation satisfying the conditions

$$s^{\perp\perp} = s,$$

$$\left(\bigvee s_i\right)^{\perp} = \bigwedge s_i^{\perp},$$

$$s \vee s^{\perp} = 1_S,$$

$$s \wedge s^{\perp} = 0_S,$$

for all $s \in S$ and $s_i \in S$. For any Gelfand quantale Q, it may be remarked that writing, for any $a \in \mathrm{R}(Q)$,

$$a^{\perp} = \bigvee_{a^* \& b = 0} b,$$

taken over all $b \in \mathrm{R}(Q)$, defines ([10], cf. [13]) an operation on the sup-lattice $\mathrm{R}(Q)$ of right-sided elements of Q that is a *pseudo-orthocomplement*, in the sense that it satisfies the conditions that

$$a \le a^{\perp\perp},$$

$$\left(\bigvee a_i\right)^{\perp} = \bigwedge a_i^{\perp},$$

$$a \wedge a^{\perp} = 0_Q,$$

for all $a \in \mathrm{R}(Q)$ and all $a_i \in \mathrm{R}(Q)$. It may be observed that this pseudo-orthocomplement is actually an orthocomplement exactly if the condition

$$a = a^{\perp\perp}$$

for all $a \in \mathrm{R}(Q)$ is satisfied, for in this situation the remaining conditions necessarily imply that

$$a \vee a^{\perp} = 1_Q$$

for all $a \in \mathrm{R}(Q)$. Noting that the sup-lattice $\mathrm{L}(Q)$ of left-sided elements of Q similarly carries a pseudo-orthocomplement, obtained by writing

$$^{\perp}a = \bigvee_{b \& a^* = 0} b$$

for any $a \in \mathrm{L}(Q)$, we make the following:

Definition 6.2. A Gelfand quantale Q is said to be a *von Neumann quantale* provided that

$$a = a^{\perp\perp}$$

for each right-sided element $a \in \mathrm{R}(Q)$ of the quantale Q (or equivalently, provided that

$$a = {}^{\perp\perp}a$$

for each left-sided element $a \in \mathrm{L}(Q)$ of Q).

A von Neumann quantale Q is therefore exactly a Gelfand quantale for which the right-sided (equivalently, left-sided) elements form an orthocomplemented sup-lattice with respect to the right (respectively, left) pseudo-orthocomplement of the quantale Q.

Given the quantale $\mathcal{Q}(S)$ of any orthocomplemented sup-lattice S, one may define [10] an involutive mapping * on the quantale by assigning to each sup-preserving mapping $\alpha \in \mathcal{Q}(S)$ the mapping defined by

$$s\alpha^* = \left(\bigvee_{t\alpha \leq s^{\perp}} t \right)^{\perp}$$

for each $s \in S$ induced by the orthocomplementation on the sup-lattice S. The mapping $\alpha^* : S \to S$ may be shown to be again sup-preserving, and the assignment thereby defined to determine an involution on the quantale $\mathcal{Q}(S)$ with respect to which it is a Gelfand quantale. Indeed, it may be remarked [10] that applying this description of the mapping * to a pseudo-orthocomplemented sup-lattice yields an involution exactly if the sup-lattice is in fact orthocomplemented.

Moreover, there are isomorphisms

$$\lambda_S : S^{\mathrm{op}} \to \mathrm{R}(\mathcal{Q}(S)),$$

$$\kappa_S : S \to \mathrm{L}(\mathcal{Q}(S)),$$

of sup-lattices, respectively from the dual of the sup-lattice S into the sup-lattice of right-sided elements of $\mathcal{Q}(S)$, and from S itself into the sup-lattice of left-sided elements of $\mathcal{Q}(S)$, defined by assigning to each $t \in S$ the sup-preserving mappings $\lambda_t, \kappa_t : S \to S$ given by

$$s\lambda_t = \begin{cases} 1_S & \text{unless} \\ 0_S & s \leq t, \end{cases}$$

and

$$s\kappa_t = \begin{cases} t & \text{unless} \\ 0_S & s = 0_S \end{cases}$$

for each $s \in S$. It may be verified straightforwardly [10] that these isomorphisms transform the orthocomplement of the sup-lattice S into respectively the right and the left pseudo-orthocomplement of the quantale $\mathcal{Q}(S)$. It follows that in fact these pseudo-orthocomplements are actually orthocomplements, and the mappings

$$\lambda_S : S^{\mathrm{op}} \to \mathrm{R}(\mathcal{Q}(S)),$$

$$\kappa_S : S \to \mathrm{L}(\mathcal{Q}(S))$$

are isomorphisms of orthocomplemented sup-lattices. In particular, the right-sided and the left-sided elements of the quantale $\mathcal{Q}(S)$ are respectively those of the form λ_t and κ_t for some element $t \in S$.

One consequence, which has some significance in the present context, is that the locale $\mathrm{I}(\mathcal{Q}(S))$ of two-sided elements of the quantale $\mathcal{Q}(S)$ is canonically isomorphic to the Boolean algebra **2**, since the only sup-preserving mappings on the orthocomplemented sup-lattice S that are of both these forms are $0_{\mathcal{Q}(S)}$, which is equal both to κ_{0_S} and to λ_{1_S}, and $1_{\mathcal{Q}(S)}$, which is both κ_{1_S} and λ_{0_S}. The Hilbert quantale $\mathcal{Q}(S)$ may therefore be considered to be a kind of quantised extension of the Boolean algebra **2**, which in the case of locales determines the concept of point.

Applying these remarks to an arbitrary Hilbert quantale, one may conclude that any Hilbert quantale is a von Neumann quantale. Conversely, it may be proved [10] that for any von Neumann quantale Q, the mapping

$$\mu_Q : Q \to \mathcal{Q}(\mathrm{R}(Q))$$

defined by assigning to each $a \in Q$ the sup-preserving mapping from $\mathrm{R}(Q)$ to itself which maps each $b \in \mathrm{R}(Q)$ to the element $a^* \& b \in \mathrm{R}(Q)$ is an isomorphism of Gelfand quantales exactly if the quantale Q is a Hilbert quantale. The Hilbert quantales may therefore be characterised intrinsically within the category of von Neumann quantales, and hence of Gelfand quantales.

Definition 6.3. By a *representation* of a Gelfand quantale Q on an orthocomplemented sup-lattice S will be meant a homomorphism

$$\varphi : Q \to \mathscr{Q}(S)$$

of Gelfand quantales from the quantale Q to the quantale $\mathscr{Q}(S)$ of the orthocomplemented sup-lattice S.

Within this context, the observations of the preceding section may be expressed by the following:

Theorem 6.1. *Any representation*

$$\varphi : A \to \mathscr{B}(H)$$

of a C^**-algebra A on a Hilbert space H determines a representation*

$$\varphi_{\mathrm{Max}\,A} : \mathrm{Max}\,A \to \mathscr{Q}(H)$$

of its spectrum $\mathrm{Max}\,A$ *on the orthocomplemented sup-lattice $\mathscr{P}(H)$ of closed linear subspaces of the Hilbert space H.*

The representation of the C^*-algebra A by a homomorphism into the C^*-algebra $\mathscr{B}(H)$ of bounded linear operators on the Hilbert space H has been reflected to a representation of the Gelfand quantale $\mathrm{Max}\,A$ by a homomorphism into the Hilbert quantale $\mathscr{Q}(H)$ of sup-preserving mappings on the orthocomplemented sup-lattice $\mathscr{P}(H)$ of closed linear subspaces of the Hilbert space H. It may be remarked further in this context that the sup-lattice $\mathscr{P}(H)$ may be identified with that of projections on the Hilbert space H. Moreover, the sup-lattice $\mathscr{P}(H)$ is an atomic orthocomplemented sup-lattice, with atoms the one-dimensional linear subspaces of the Hilbert space H. Now, suppose that an irreducible representation

$$\varphi : A \to \mathscr{B}(H)$$

is given. For any non-zero element $x \in H$ of the Hilbert space H, consider the closed linear subspace $\langle x \rangle \in \mathscr{P}(H)$ generated by $x \in H$, which is therefore an atom of the sup-lattice $\mathscr{P}(H)$. Denote by

$$M_x \in \mathscr{Q}(H)$$

the right-sided element of $\mathscr{Q}(H)$ obtained by applying $\lambda_H : \mathscr{P}(H)^{\mathrm{op}} \to \mathrm{R}(\mathscr{Q}(H))$ to the element $\langle x \rangle \in \mathscr{P}(H)$. Since $\langle x \rangle \in \mathscr{P}(H)$ is an atom, that is, a minimal element, of $\mathscr{P}(H)$, the element $M_x \in \mathscr{Q}(H)$ will be a maximal element of the right side of $\mathscr{Q}(H)$.

Recalling [4] that the maximal right ideal $\mathfrak{m}_x$ of the C^*-algebra A determined by the irreducible representation

$$\varphi : A \to \mathscr{B}(H)$$

together with the non-zero element $x \in H$ is obtained by taking

$$\mathfrak{m}_x = \{a \in A \mid x\varphi_a = 0_H\},$$

we assert the following:

Corollary 6.2. *For any irreducible representation*

$$\varphi : A \to \mathscr{B}(H)$$

of a C*-*algebra A on a Hilbert space H, the maximal right ideal $\mathfrak{m}_x$ of A determined by a non-zero element $x \in H$ is that obtained by taking the closed linear subspace*

$$\mathfrak{m}_x = \bigvee_{N\varphi_{\mathrm{Max}\,A} \le M_x} N$$

of the C*-*algebra A corresponding to the maximal right-sided element $M_x \in \mathscr{Q}(H)$ of the Hilbert quantale $\mathscr{Q}(H)$.*

Moreover, any maximal right ideal of the C*-*algebra A yielding an irreducible representation of A equivalent to this representation on H is of this form for some non-zero element $x \in H$.*

Proof. From the expression

$$\mathfrak{m}_x = \{a \in A \mid x\varphi_a = 0_H\}$$

already noted, we observe that this may be rewritten in terms of the representation of the quantale $\mathrm{Max}\,A$ on the orthocomplemented sup-lattice $\mathscr{P}(H)$ in the form

$$\mathfrak{m}_x = \bigvee_{\langle x\rangle\varphi_a \le \langle 0_H\rangle} \langle a\rangle$$

in which now $\varphi_a : \mathscr{P}(H) \to \mathscr{P}(H)$ denotes the sup-preserving mapping assigned to the closed linear subspace $\langle a\rangle \in \mathrm{Max}\,A$.

Now, we assert that the condition $\langle x\rangle\varphi_a \le 0_{\mathscr{P}(H)}$ describing the extent of the supremum is equivalent to requiring that $\varphi_a \le \lambda_{\langle x\rangle}$ in the quantale $\mathscr{Q}(H)$. For this latter inequality corresponds to asking that

$$M\varphi_a \le M\lambda_{\langle x\rangle}$$

for all closed linear subspaces M of the Hilbert space H. By the description of the element $\lambda_{\langle x\rangle}$ of the right side of $\mathscr{Q}(H)$, this is equivalent to requiring that

$$M\varphi_a \le \begin{cases} H & \text{unless} \\ \{0_H\} & M \le \langle x\rangle. \end{cases}$$

However, this requirement for each closed linear subspace M reduces to the condition that

$$\langle x\rangle\varphi_a \le \langle 0_H\rangle,$$

since it is trivially satisfied for any other closed linear subspace M, as asserted.

Applying this to the description of the required supremum, we have that

$$\mathfrak{m}_x = \bigvee_{\varphi_a \le \lambda_{\langle x\rangle}} \langle a\rangle.$$

Observing that φ_a denotes the image under the representation

$$\varphi_{\mathrm{Max}\,A} : \mathrm{Max}\,A \to \mathscr{Q}(H)$$

of the closed linear subspace $\langle a \rangle$ of the C*-algebra A, and that in the present context we have chosen to denote the maximal right-sided element $\lambda_{\langle x \rangle}$ of $\mathscr{Q}(H)$ by M_x, this expression may be rewritten in the form

$$\mathfrak{m}_x = \bigvee_{\langle a \rangle \varphi_{\operatorname{Max} A} \leq M_x} \langle a \rangle.$$

However, the quantale $\operatorname{Max} A$, as a sup-lattice, is also atomic, since any closed linear subspace of the C*-algebra A is the supremum of the subspaces $\langle a \rangle \in \operatorname{Max} A$ for each $a \in A$ that it contains. Hence, the maximal right ideal $\mathfrak{m}_x$ may indeed be written in the form

$$\mathfrak{m}_x = \bigvee_{N \varphi_{\operatorname{Max} A} \leq M_x} N$$

in which N ranges over the closed linear subspaces of the C*-algebra A.

The final assertion of the corollary simply reflects the observation already made, that indeed every maximal right ideal of the C*-algebra A that yields a representation equivalent to that given is of the form $\mathfrak{m}_x$ for some non-zero element $x \in H$, which completes the proof. □

The importance of the corollary lies in the form of the expression

$$\mathfrak{m}_x = \bigvee_{N \varphi_{\operatorname{Max} A} \leq M_x} N$$

for the maximal right ideal determined by the non-zero element $x \in H$. For the representation

$$\varphi_{\operatorname{Max} A} : \operatorname{Max} A \to \mathscr{Q}(H)$$

is actually the inverse image mapping

$$\varphi^* : \operatorname{Max} A \to \mathscr{Q}(H)$$

of a map in the category of Gelfand quantales, where for the sake of clarity we drop the subscript $\operatorname{Max} A$, hence admits a direct image mapping

$$\varphi_* : \mathscr{Q}(H) \to \operatorname{Max} A,$$

constructed by taking the coadjoint of the sup-preserving mapping denoted by φ^*. Applying the construction of this coadjoint to evaluate this direct image mapping at the element M_x of the quantale $\mathscr{Q}(H)$, one obtains that

$$M_x \, \varphi_* = \bigvee_{N \; \varphi_{\operatorname{Max} A} \leq M_x} N.$$

The maximal right ideal $\mathfrak{m}_x$ of the C*-algebra A obtained from the irreducible representation

$$\varphi : A \to \mathscr{B}(H)$$

by the non-zero element $x \in H$ is therefore exactly the direct image of the maximal right-sided element M_x of the quantale $\mathscr{Q}(H)$ determined by the non-zero element $x \in H$.

7. Pure states

The concept of a pure state of a C*-algebra A has already been met with in describing the inner product structure on the Hilbert space $A/\mathfrak{m}$ determined by a maximal right ideal $\mathfrak{m}$. In that context, it was remarked that the pure state

$$\pi_{\mathfrak{m}} : A \to \mathbb{C}$$

corresponding to the maximal right ideal $\mathfrak{m}$ has kernel a closed linear subspace $\mathfrak{n}$ of the C*-algebra A that is self-adjoint, and is the unique proper self-adjoint closed linear subspace that contains the maximal right ideal $\mathfrak{m}$. Furthermore, the maximal right ideal $\mathfrak{m}$ may be recovered from the self-adjoint closed linear subspace $\mathfrak{n}$ as the largest closed right ideal contained in $\mathfrak{n}$.

In this section, we apply these and other observations about the pure states of a C*-algebra to motivate the consideration of pure states in the context of Gelfand quantales, showing that the pure states of a C*-algebra A may be identified with the pure states of its spectrum $\operatorname{Max} A$, and characterising the pure states of a Hilbert quantale $\mathcal{Q}(S)$ in the case that the the orthocomplemented sup-lattice S is atomic.

With these remarks, we make the following:

Definition 7.1. By a pure state $\mathfrak{n}$ of a Gelfand quantale Q will be meant a proper self-adjoint element $\mathfrak{n} \in Q$ with the property that it is the unique proper self-adjoint element of Q that contains the largest right-sided element $\mathfrak{m} \in Q$ contained in $\mathfrak{n} \in Q$.

By the observations made above, the kernel of any pure state

$$\pi : A \to \mathbb{C}$$

of a C*-algebra A is a pure state of the spectrum $\operatorname{Max} A$ of the C*-algebra. Moreover, any pure state $\mathfrak{n}$ of the spectrum of the C*-algebra A is determined in this way, by considering the pure state

$$\pi_{\mathfrak{m}} : A \to \mathbb{C}$$

determined by the largest closed right ideal of A that is contained in the pure state $\mathfrak{n}$ of the Gelfand quantale $\operatorname{Max} A$. For, since the pure state $\mathfrak{n}$ is a proper self-adjoint element of the quantale $\operatorname{Max} A$, the closed right ideal $\mathfrak{m}$ is also proper, hence contained in a maximal right ideal of A. The pure state corresponding to this maximal right ideal of A is therefore necessarily the unique proper self-adjoint element of $\operatorname{Max} A$ containing the closed right ideal $\mathfrak{m}$, hence equals the pure state $\mathfrak{n}$ of the quantale $\operatorname{Max} A$. In turn, it may be seen that the closed right ideal $\mathfrak{m}$ is necessarily the maximal right ideal corresponding to this pure state of the C*-algebra A.

In this way, the pure states of the C*-algebra A may indeed be seen straightforwardly to correspond bijectively with the pure states of its spectrum $\operatorname{Max} A$. In particular, any maximal right ideal of the C*-algebra A is contained in a unique pure state of the spectrum $\operatorname{Max} A$, and any pure state of the spectrum $\operatorname{Max} A$ contains a unique maximal right ideal of the C*-algebra A. In the case of a Hilbert quantale $\mathcal{Q}(S)$, a similar situation holds, provided that we ensure that there exist the maximal right-sided elements that are needed to provide the correspondence. Since, by the canonical isomorphism

$$\lambda_S : S^{\mathrm{op}} \to \mathrm{R}(\mathcal{Q}(S)),$$

the maximal right-sided elements of the Hilbert quantale $\mathcal{Q}(S)$ correspond to the atoms of the orthocomplemented sup-lattice S, we recall the following:

Definition 7.2. An orthocomplemented sup-lattice S is said to be *atomic* provided that each element $s \in S$ is equal to the join

$$\bigvee_{x \leq s} x$$

of the atoms $x \in S$ lying below it.

Of course, by the duality referred to above, the condition that the orthocomplemented sup-lattice S is atomic is exactly equivalent to the requirement that any right-sided element of the Hilbert quantale $\mathcal{Q}(S)$ is the meet of the maximal right-sided elements containing it, as is the case for the spectrum $\mathrm{Max}\,A$ of a C*-algebra A.

With these preliminaries, we observe that the observations concerning the pure states of a C*-algebra A extend to the case of the Hilbert quantale $\mathcal{Q}(S)$ determined by an atomic orthocomplemented sup-lattice S:

Theorem 7.1. *Any maximal right-sided element M of the Hilbert quantale $\mathcal{Q}(S)$ of an atomic orthocomplemented sup-lattice S is contained in a unique pure state N, obtained by taking*

$$N = M \vee M^*.$$

Any pure state N of the Hilbert quantale $\mathcal{Q}(S)$ contains a unique maximal right-sided element M, obtained by taking the join of all right-sided elements contained in N.

In particular, for any Hilbert quantale $\mathcal{Q}(S)$ determined by an atomic orthocomplemented sup-lattice S, there is a bijective correspondence between the pure states of $\mathcal{Q}(S)$ and the maximal right-sided elements of $\mathcal{Q}(S)$.

Proof. In fact, we shall prove rather more concerning the quantale $\mathcal{Q}(S)$ while proving the assertions of the theorem. Firstly, we recall that the canonical isomorphism

$$\lambda_S : S^{\mathrm{op}} \to \mathrm{R}(\mathcal{Q}(S))$$

from the dual of the sup-lattice S to the sup-lattice of right-sided elements of $\mathcal{Q}(S)$ was defined by assigning to each $t \in S$ the sup-preserving mapping $\lambda_t : S \to S$ given by

$$s\lambda_t = \begin{cases} 1_S & \text{unless} \\ 0_S & s \leq t \end{cases}$$

for each $s \in S$. It is in this way that any maximal right-sided element of the Hilbert quantale $\mathcal{Q}(S)$ is therefore the image of an atom $x \in S$ of the sup-lattice S. The maximal right-sided element of $\mathscr{Q}(S)$ determined by an atom $x \in S$ will be denoted by M_x, and is therefore given by the description

$$sM_x = \begin{cases} 1_S & \text{unless} \\ 0_S & s \leq x \end{cases}$$

for each $s \in S$. Given a maximal right-sided element $M_x \in \mathscr{Q}(S)$, observe that the elements of $\mathscr{Q}(S)$ that lie in the up-segment of the element M_x are exactly those of the form $\tau_s : S \to S$ for any $s \in S$, defined by setting

$$t\tau_s = \begin{cases} 1_S & \text{unless} \\ s & t = x \\ 0_S & t = 0_S \end{cases}$$

for each $t \in S$. In particular, such an element is entirely determined by its value at the given element $x \in S$. Indeed, with respect to the canonical pointwise ordering on the quantale $\mathscr{Q}(S)$, it is evident that the mapping

$$\uparrow M_x \to S$$

induced by evaluation at $x \in S$ is in fact an isomorphism of sup-lattices from the up-segment of the maximal right-sided element M_x to the sup-lattice S.

Evidently, an element $\alpha \geq M_x$ is proper exactly if its value $x\alpha \in S$ is proper in the sup-lattice S. Moreover, recalling the observation that the involution of the Hilbert quantale $\mathscr{Q}(S)$ assigns to the element M_x that is the image of $x \in S$ under the canonical dual isomorphism

$$\lambda_S : S^{\mathrm{op}} \to \mathrm{R}(\mathscr{Q}(S))$$

the element M_x^* that is the image of $x^\perp \in S$ under the canonical isomorphism

$$\kappa_S : S \to \mathrm{L}(\mathscr{Q}(S)),$$

it follows from the fact that $M_x \leq \alpha$ implies $M_x^* \leq \alpha^*$ that any self-adjoint α lying above the element M_x necessarily satisfies the condition that

$$x^\perp \leq x\alpha.$$

Since $x^\perp \in S$ is maximal in the sup-lattice S in the event that $x \in S$ is an atom of S, it follows either that $x\alpha = x^\perp$, in which case α is exactly the element $\tau_{x^\perp} \in \mathscr{Q}(S)$, or that $x\alpha = 1_S$, in which case α is the top element $1_{\mathscr{Q}(S)} \in \mathscr{Q}(S)$. It may be verified straightforwardly that $\tau_{x^\perp} \in \mathscr{Q}(S)$ is indeed a proper self-adjoint element of $\mathscr{Q}(S)$ that contains the maximal right-sided element $M_x \in \mathscr{Q}(S)$, hence is necessarily unique by the above remarks. It follows in fact that this pure state corresponding to the maximal right-sided element $M_x \in \mathscr{Q}(S)$, which we shall denote by $N_x \in \mathscr{Q}(S)$, is given implicitly by taking the join

$$N_x = M_x \vee M_x^*$$

of M_x with its involute M_x^* in the quantale $\mathscr{Q}(S)$, and explicitly by the description

$$sN_x = \begin{cases} 1_S & \text{unless} \\ x^\perp & t = x \\ 0_S & t = 0_S \end{cases}$$

for each $s \in S$. In particular, any maximal right-sided element M_x of the Hilbert quantale $\mathscr{Q}(S)$ is contained in a unique pure state N_x of the quantale $\mathscr{Q}(S)$, as asserted.

Conversely, and using for the first time the atomicity of the orthocomplemented sup-lattice S, given a pure state N of the Hilbert quantale $\mathscr{Q}(S)$ let M denote the join of all the right-sided elements of $\mathscr{Q}(S)$ that are contained in N. Since M is necessarily proper, we may choose a maximal right-sided element M_x of the Hilbert quantale that contains M, by applying the atomicity of the orthocomplemented sup-lattice S. (Explicitly, we take $x \in S$ to be an atom lying below the element of S that is associated with the right-sided element M under the dual isomorphism from S^{op} to $\mathrm{R}(\mathscr{Q}(S))$.)

Now, by the remarks above, there is a unique pure state of $\mathscr{Q}(S)$ that contains the maximal right-sided element M_x, namely the element N_x. Since N_x is then a proper self-adjoint element of $\mathscr{Q}(S)$ that contains M_x and hence M, by the choice of M_x, it follows by the uniqueness of N amongst proper self-adjoint elements containing M that it coincides with the pure state N_x. Of course, it then follows that M is exactly the maximal right-sided element M_x, and is the unique maximal right-sided element contained in the pure state N.

In particular, there is evidently a bijective correspondence between the pure states of the Hilbert quantale $\mathscr{Q}(S)$, and the maximal right-sided elements of the Hilbert quantale $\mathscr{Q}(S)$ (and also with the atoms of the orthocomplemented sup-lattice S), which completes the proof of the theorem. □

In proving the theorem, we have observed that for any atom $x \in S$ of the orthocomplemented sup-lattice S there is a canonical isomorphism

$$\uparrow M_x \to S$$

from the up-segment of the maximal right-sided element of the Hilbert quantale $\mathscr{Q}(S)$ determined by $x \in S$ to the sup-lattice S. It may further be shown that, with respect to a canonical orthocomplement on the sup-lattice $\uparrow M_x$ this is indeed an isomorphism of orthocomplemented sup-lattices. Before describing this orthocomplement on the up-segment of the maximal right-sided element M_x of the Hilbert quantale $\mathscr{Q}(S)$, we examine the corresponding situation in the case of the spectrum $\mathrm{Max}\, A$ of a C^*-algebra A.

It has been remarked already that, for the maximal right ideal $\mathfrak{m}_x$ obtained from an irreducible representation

$$\varphi : A \to \mathscr{B}(H)$$

of the C^*-algebra A on a Hilbert space H by choosing a non-zero element $x \in H$, and hence an atom of the orthocomplemented sup-lattice $\mathscr{P}(H)$ of closed linear subspaces of H, the quotient space $A/\mathfrak{m}_x$ is canonically a Hilbert space, on which A admits by right multiplication an irreducible representation

$$\varphi_x : A \to \mathscr{B}(A/\mathfrak{m}_x)$$

that is equivalent to the given representation. In particular, there is a canonical isomorphism

$$A/\mathfrak{m}_x \to H$$

of Hilbert spaces that induces this equivalence. The inner product on the quotient space $A/\mathfrak{m}_x$ with respect to which this isomorphism of Hilbert spaces exists is that obtained from the pure state

$$\pi_x : A \to \mathbb{C}$$

associated with this irreducible representation by setting

$$\langle a + \mathfrak{m}_x, b + \mathfrak{m}_x \rangle = \pi_x(ab^*)$$

for each $a, b \in A$. In particular, the relation of orthogonality on the quotient space $A/\mathfrak{m}_x$ which is determined by, and which, in conjunction with the quotient norm, determines, this inner product is that given by writing

$$a + \mathfrak{m}_x \perp b + \mathfrak{m}_x \quad \text{if, and only if,} \quad ab^* \in \mathfrak{n}_x,$$

in which $\mathfrak{n}_x$ denotes the kernel of the pure state, hence the pure state of the spectrum $\operatorname{Max} A$ of the C*-algebra A corresponding to the maximal right ideal $\mathfrak{m}_x$.

Observing that the sup-lattice $\mathscr{P}(A/\mathfrak{m}_x)$ of closed linear subspaces of the Hilbert space $A/\mathfrak{m}_x$ thereby determined is isomorphic to the up-segment $\uparrow\mathfrak{m}_x$ of the maximal right ideal $\mathfrak{m}_x$ in the spectrum $\operatorname{Max} A$ of the C*-algebra A, there is therefore a canonical isomorphism

$$\uparrow\mathfrak{m}_x \to \mathscr{P}(H)$$

of sup-lattices, which is moreover an isomorphism of orthocomplemented sup-lattices with respect to the orthocomplement on $\uparrow\mathfrak{m}_x$ defined by the orthogonality relation given by setting

$$M \perp N \quad \text{if, and only if,} \quad M \& N^* \leq \mathfrak{n}_x$$

for each $M, N \geq \mathfrak{m}_x$ in the quantale $\operatorname{Max} A$.

The pure state $\mathfrak{n}_x$ of the spectrum $\operatorname{Max} A$ corresponding to the maximal right ideal $\mathfrak{m}_x$ therefore provides the means of describing a representation equivalent to the given representation

$$\varphi_{\operatorname{Max} A} : \operatorname{Max} A \to \mathscr{Q}(H)$$

on the Hilbert space H in terms that are intrinsic to the quantale $\operatorname{Max} A$. Moreover, it may be verified straightforwardly that the image of the pure state N_x of the Hilbert quantale $\mathscr{Q}(H)$ under the direct image mapping of the representation is exactly the pure state $\mathfrak{n}_x$ of the spectrum $\operatorname{Max} A$ corresponding to the maximal right ideal $\mathfrak{m}_x$ which is the image of the maximal right-sided element M_x of $\mathscr{Q}(H)$ under this direct image mapping. In particular, allowing the choice of atom $x \in \mathscr{P}(H)$ of the orthocomplemented sup-lattice $\mathscr{P}(H)$ to vary, we see that each pure state of the Hilbert quantale $\mathscr{Q}(H)$ maps under the direct image mapping to a pure state of the spectrum $\mathrm{Max}\,A$, a condition to which we shall later make reference.

Concerning the case of the Hilbert quantale $\mathscr{Q}(S)$ determined by an arbitrary atomic orthocomplemented sup-lattice S, we have the following:

Corollary 7.2. *For any maximal right-sided element M_x of the Hilbert quantale $\mathscr{Q}(S)$ determined by an orthocomplemented sup-lattice S, the canonical isomorphism*

$$\uparrow M_x \to S$$

from the up-segment of M_x to the sup-lattice is an isomorphism of orthocomplemented sup-lattices with respect to the orthocomplement defined by writing

$$M \perp N \quad \textit{if, and only if,} \quad M \& N^* \leq N_x$$

for each $M, N \geq M_x$.

Proof. Recalling that any $M, N \in \mathscr{Q}(S)$ lying in the up-segment of M_x may be written in the form τ_s, τ_t for some elements $s, t \in S$, it suffices to establish that

$$\tau_s \& \tau_t^* \leq N_x \quad \text{if, and only if,} \quad s \leq t^{\perp}.$$

But, $\tau_s \& \tau_t^* \leq N_x$ if, and only if, $(x\,\tau_s)\tau_t^* \leq x^{\perp}$, by the definition of the pure state N_x and the pointwise ordering of the Hilbert quantale $\mathscr{Q}(S)$. Since $x\tau_s = s$, by the definition of this element of $\mathscr{Q}(S)$, this is equivalent to requiring that $s\tau_t^* \leq x^{\perp}$, which by the description of the involution on $\mathscr{Q}(S)$, is equivalent to the condition that $x\tau_t \leq s^{\perp}$. Again by the definition of τ_t, this is equivalent to asking that $t \leq s^{\perp}$, or equivalently that $s \leq t^{\perp}$, as required. The canonical isomorphism therefore preserves and reflects orthogonality, hence, with respect to the orthocomplement thereby defined in terms of the orthogonality relation described above on the up-segment of M_x, the canonical isomorphism is indeed an isomorphism of orthocomplemented sup-lattices as asserted. □

It is therefore possible to recover the orthocomplemented sup-lattice underlying a Hilbert quantale by considering any maximal right-sided element M_x of the quantale, together with the pure state N_x that it determines. In the event that the orthocomplemented sup-lattice is atomic, then the existence of such maximal right-sided elements is assured, with the up-segment of each and every one of them being canonically isomorphic to the orthocomplemented sup-lattice concerned. In particular, the pure states of a Hilbert quantale $\mathscr{Q}(S)$ are seen to play a role with respect to orthocomplementation that is analogous to that played by the pure states of a C*-algebra A in the case of its spectrum, an observation that we shall now proceed to exploit.

8. Irreducible representations

At this point, we introduce the concept of the irreducibility of a representation

$$\varphi : \operatorname{Max} A \to \mathscr{Q}(S)$$

of the spectrum $\operatorname{Max} A$ of a C*-algebra A on an orthocomplemented sup-lattice S, following closely that of a representation of a C*-algebra on a Hilbert space. It may be recalled that a representation

$$\varphi : A \to \mathscr{B}(H)$$

of a C*-algebra A on a Hilbert space H is said to be irreducible provided that the only closed linear subspaces of the Hilbert space H that are invariant under the action of the C*-algebra A are the zero subspace and H itself, motivating the following:

Definition 8.1. A representation

$$\varphi : \operatorname{Max} A \to \mathscr{Q}(S)$$

of the spectrum $\operatorname{Max} A$ of a C*-algebra A on an atomic orthocomplemented sup-lattice S will be said to be *irreducible* provided that it is non-zero, and that

$$s\,\varphi_M \le s \quad \text{for all } M \in \operatorname{Max} A \quad \text{implies } s = 0_S \text{ or } s = 1_S$$

for any element $s \in S$, in which $\varphi_M \in \mathscr{Q}(S)$ denotes the image of the closed linear subspace M of the C*-algebra A under the representation.

The representation

$$\varphi_{\operatorname{Max} A} : \operatorname{Max} A \to \mathscr{Q}(H)$$

of the spectrum $\operatorname{Max} A$ of the C*-algebra A on the orthocomplemented sup-lattice $\mathscr{P}(H)$ of closed linear subspaces of the Hilbert space H determined by an irreducible representation

$$\varphi : A \to \mathscr{B}(H)$$

of the C*-algebra A on the Hilbert space H is evidently irreducible in the present sense, by the observation that the orthocomplemented sup-lattice $\mathscr{P}(H)$ is just that of the closed linear subspaces of the Hilbert space H.

Within the context of quantales, however, we have the following characterisation of the irreducibility of a representation:

Theorem 8.1. *For any representation*

$$\varphi : \operatorname{Max} A \to \mathscr{Q}(S)$$

of the spectrum $\operatorname{Max} A$ *of a* C*-*algebra* A *on an atomic orthocomplemented sup-lattice* S, *a necessary and sufficient condition that the representation be irreducible is that*

$$1_{\operatorname{Max} A}\varphi^* = 1_{\mathscr{Q}(S)}.$$

Proof. For, recalling that the top element of $\mathscr{Q}(S)$ is the mapping defined by

$$s1_{\mathscr{Q}(S)} = \begin{cases} 1_S & \text{unless} \\ 0_S & s = 0 \end{cases}$$

for any $s \in S$, it follows that the condition that the top element of $\operatorname{Max} A$ maps to that of the Hilbert quantale $\mathscr{Q}(S)$ is exactly that

$$s\varphi_A = 1_S$$

for any non-zero $s \in S$. If the representation is irreducible, then consider the element

$$s\varphi_A \in S$$

determined by any non-zero element $s \in S$. Evidently, this element is invariant with respect to the representation, since

$$(s\varphi_A)\,\varphi_M \ \leq \ s\varphi_A$$

for any closed linear subspace $M \in \operatorname{Max} A$, since necessarily $A\&M \leq A\&A \leq A$. Hence, by the irreducibility of the representation, we may conclude that $s\varphi_A = 0_S$ or $s\varphi_A = 1_S$. Observing that $s \in S$ being non-zero implies that $s\varphi_A \in S$ is non-zero, since necessarily $s \leq s\varphi_A$ since the C*-algebra is supposed unital, it follows that $s\varphi_A = 1_S$. Thus, we have that

$$1_{\operatorname{Max} A}\varphi^* = 1_{\mathscr{Q}(S)},$$

as asserted. Conversely, if the representation

$$\varphi : \operatorname{Max} A \to \mathscr{Q}(S)$$

maps the top element of $\operatorname{Max} A$ to the top element of the Hilbert quantale $\mathscr{Q}(S)$, then the representation is necessarily irreducible. For, given any element $s \in S$ for which

$$s\,\varphi_M \leq s$$

for all closed linear subspaces $M \in \operatorname{Max} A$, we assert that $s = 0_A$ or $s = 1_S$. For, if $s \in S$ is non-zero, then by the condition assumed of the representation we have that

$$\varphi_A = 1_S,$$

by the definition of the top element of the Hilbert quantale $\mathscr{Q}(S)$. Hence,

$$s = 1_S,$$

since in particular

$$s\,\varphi_A \ \leq \ s,$$

by the invariance of the given element $s \in S$. The representation is therefore necessarily irreducible, as asserted, which completes the proof of the theorem. □

It may be remarked that since any irreducible representation

$$\varphi : \operatorname{Max} A \to \mathscr{Q}(S)$$

of the spectrum of a C*-algebra A on a Hilbert quantale $\mathcal{Q}(S)$ maps the top element of $\operatorname{Max} A$ to that of the Hilbert quantale $\mathcal{Q}(S)$, it follows that any closed right ideal I of the C*-algebra A is mapped by the representation to a right-sided element of the Hilbert quantale $\mathcal{Q}(S)$. For, given any element $I \in R(\operatorname{Max} A)$ of the right-side of the spectrum $\operatorname{Max} A$, one has that

$$I \& A \leq I \quad \text{implies} \quad \varphi_I \& \varphi_A \leq \varphi_I.$$

Observing that φ_A is necessarily the top element $1_{\mathcal{Q}(S)}$ of the Hilbert quantale $\mathcal{Q}(S)$, by the irreducibility of the representation, we conclude that indeed

$$\varphi_I \& 1_{\mathcal{Q}(S)} \leq \varphi_I,$$

as asserted. Hence, any right-sided element of $\operatorname{Max} A$ is mapped to a right-sided element of $\mathcal{Q}(S)$. Indeed, as the converse may straightforwardly be proved, this condition is again equivalent to that of the irreducibility of the representation. A representation

$$\varphi : \operatorname{Max} A \to \mathcal{Q}(S)$$

is therefore irreducible exactly if it restricts to a homomorphism

$$\mathrm{R}(\varphi) : \mathrm{R}(\operatorname{Max} A) \to \mathrm{R}(\mathcal{Q}(S))$$

of the corresponding quantales of right-sided elements, an observation to which we shall later return.

Adjointly, the irreducibility of a representation

$$\varphi : \operatorname{Max} A \to \mathcal{Q}(S)$$

implies that the direct image mapping

$$\varphi_* : \mathcal{Q}(S) \to \operatorname{Max} A$$

also takes right-sided elements of $\mathcal{Q}(S)$ into closed right ideals of the C*-algebra A, since given any right-sided element $M \in \mathcal{Q}(S)$ one has that

$$M\varphi_*\varphi^* \& 1_{\mathcal{Q}(S)} \leq M \& 1_{\mathcal{Q}(S)} \leq M$$

by the coadjunction relation $M\varphi_*\varphi^* \leq M$ and the right-sidedness of M. By the observations above concerning irreducibility, and by the fact that the inverse image mapping preserves products, it follows that

$$(M\varphi_* \& 1_{\operatorname{Max} A})\,\varphi^* \leq M,$$

from which by adjointness, we may conclude that

$$M\varphi_* \& 1_{\operatorname{Max} A} \leq M\varphi_*,$$

as asserted. In other words, the direct image mapping induced by the representation also restricts to a direct image mapping

$$\mathrm{R}(\varphi_*) : \mathrm{R}(\mathcal{Q}(S)) \to \mathrm{R}(\operatorname{Max} A),$$

coadjoint to the restriction of the inverse image mapping.

It may also be remarked in passing that the direct image mapping

$$\varphi_* : \mathcal{Q}(S) \to \operatorname{Max} A$$

necessarily also preserves involution, since for any $M \in \mathcal{Q}(S)$ one has that

$$\begin{aligned} M^*\varphi_* &= \bigvee_{N\varphi^* \le M^*} N = \bigvee_{(N\varphi^*)^* \le M} N \\ &= \bigvee_{N^*\varphi^* \le M} N = \bigvee_{N\varphi^* \le M} N^* = \left(\bigvee_{N\varphi^* \le M} N \right)^* \\ &= (M\varphi_*)^*, \end{aligned}$$

in which N denotes an arbitrary element of the quantale $\operatorname{Max} A$. In particular, this implies that the direct image mapping, as is the case for the inverse image homomorphism, evidently maps self-adjoint elements of the Hilbert quantale $\mathcal{Q}(S)$ to self-adjoint elements of the spectrum $\operatorname{Max} A$.

Before considering the implications of these observations, particularly in relation to the pure states of the quantales concerned, it should be noted in passing that we have tacitly adopted a number of notational conventions concerning a representation

$$\varphi : \operatorname{Max} A \to \mathcal{Q}(S)$$

of the spectrum $\operatorname{Max} A$ on the orthocomplemented sup-lattice S, which may be summarised as follows. When considering the image of a closed linear subspace M of the C*-algebra A as a sup-preserving mapping from S to itself, we shall tend to stay with the notation

$$\varphi_M : S \to S$$

already introduced. Whereas, when considering this image primarily with respect to the lattice-theoretic properties of the Hilbert quantale $\mathcal{Q}(S)$, we shall invoke inverse image notation to denote this element of the Hilbert quantale $\mathcal{Q}(S)$ by

$$M\varphi^* \in \mathcal{Q}(S).$$

It may also be noted in passing that the symbols M, N will be applied indiscriminately to denote elements of the spectrum $\operatorname{Max} A$ and of the Hilbert quantale $\mathcal{Q}(S)$. Concerning the direct image mapping

$$\varphi_* : \mathcal{Q}(S) \to \operatorname{Max} A,$$

the above observations may be extended to note that any pure state $N \in \mathcal{Q}(S)$ of the Hilbert quantale $\mathcal{Q}(S)$ is mapped to an element $N\varphi_* \in \operatorname{Max} A$ that is both proper and self-adjoint. For certainly, by the above remarks, the direct image of a self-adjoint element is necessarily self-adjoint. It is also proper, since the direct image of any proper element $N \in \mathcal{Q}(S)$ is necessarily proper. For, suppose that on the contrary $N\varphi_* \in \operatorname{Max} A$ were not proper. Then $1_{\operatorname{Max} A} \le N\varphi_*$ implies by adjointness that $1_{\operatorname{Max} A}\varphi^* \le N$. But, by the irreducibility of the representation, we then have that $1_{\mathcal{Q}(S)} \le N$, contradicting the properness of the element $N \in \mathcal{Q}(S)$. Hence, the direct image mapping preserves properness; in particular, the direct image mapping necessarily maps each pure state of the Hilbert quantale $\mathcal{Q}(S)$ to a proper self-adjoint element of the spectrum $\operatorname{Max} A$.

Concerning the possibility that a pure state $\mathfrak{n} \in \operatorname{Max} A$ of the spectrum $\operatorname{Max} A$ might actually be the direct image of a pure state $N \in \mathcal{Q}(S)$ of the Hilbert quantale $\mathcal{Q}(S)$, we have the following:

Corollary 8.2. *For any irreducible representation*

$$\varphi : \operatorname{Max} A \to \mathcal{Q}(S)$$

of the spectrum of a C^**-algebra A on an atomic orthocomplemented sup-lattice S, and for any pure state $\mathfrak{n}$ of the spectrum* $\operatorname{Max} A$, *the following assertions are equivalent*:

(a) *the inverse image of the pure state $\mathfrak{n}$ is a proper element of the Hilbert quantale* $\mathcal{Q}(S)$;

(b) *there exists a pure state N of the Hilbert quantale $\mathcal{Q}(S)$ of which the direct image is the pure state $\mathfrak{n}$.*

Proof. Suppose that the pure state $\mathfrak{n}$ of the spectrum $\operatorname{Max} A$ has inverse image a proper element of $\mathcal{Q}(S)$. Consider the maximal right ideal $\mathfrak{m}$ determined by the pure state $\mathfrak{n}$ of $\operatorname{Max} A$, and observe that since $\mathfrak{m} \le \mathfrak{n}$ it follows that $m\varphi^* \le n\varphi^*$, giving that the inverse image $\mathfrak{m}\varphi^*$ is also proper. Moreover, by the remarks above, the inverse image $\mathfrak{m}\varphi^*$ is right-sided in the Hilbert quantale $\mathcal{Q}(S)$, by the irreducibility of the representation. By the atomicity of the orthocomplemented sup-lattice S, there exists a maximal right-sided element M_x for which $\mathfrak{m}\varphi^* \le M_x$, in which $x \in S$ denotes the corresponding atom of S. Noting that the inverse image homomorphism preserves involution, and recalling that necessarily $\mathfrak{n} = \mathfrak{m} \vee \mathfrak{m}^*$, it follows that

$$\mathfrak{n}\varphi^* = (\mathfrak{m} \vee \mathfrak{m}^*)\varphi^* = \mathfrak{m}\varphi^* \vee (\mathfrak{m}\varphi^*)^* \le M_x \vee M_x^* = N_x.$$

Again by adjointness, $\mathfrak{n}\varphi^* \le N_x$ implies that $\mathfrak{n} \le N_x\varphi_*$, while by the remarks above we have that $N_x\varphi_*$ is necessarily self-adjoint and proper. By the maximality of a pure state amongst such elements, we may conclude that $\mathfrak{n} = N_x\varphi_*$. Hence, the pure state $\mathfrak{n}$ is indeed the direct image of a pure state of the Hilbert quantale $\mathcal{Q}(S)$.

Conversely, suppose that N is a pure state of the Hilbert quantale $\mathcal{Q}(S)$ for which $N\varphi_* = \mathfrak{n}$. Then, by adjointness, one has that $\mathfrak{n} \le N\varphi_*$ implies that $\mathfrak{n}\varphi^* \le N$. Since N is a pure state, it is necessarily proper. Hence, a fortiori the inverse image of $\mathfrak{n}$ is indeed proper. $\square$

Hence, an irreducible representation of the spectrum $\operatorname{Max} A$ of a C^*-algebra A on an atomic orthocomplemented sup-lattice S either maps every pure state $\mathfrak{n}$ of $\operatorname{Max} A$ trivially to the element $1_{\mathcal{Q}(S)} \in \mathcal{Q}(S)$ of the Hilbert quantale $\mathcal{Q}(S)$, or else satisfies the conditions of the following:

Definition 8.2. An irreducible representation

$$\varphi : \operatorname{Max} A \to \mathcal{Q}(S)$$

of the spectrum of a C*-algebra A on an orthocomplemented sup-lattice S will be said to be *non-trivial* provided that there exists a pure state $\mathfrak{n}$ of the spectrum $\operatorname{Max} A$ of which the inverse image $\mathfrak{n}\varphi^*$ is a proper element of the Hilbert quantale $\mathscr{Q}(S)$.

It may be seen straightforwardly that the representation

$$\varphi : \operatorname{Max} A \to \mathscr{Q}(H)$$

of the spectrum $\operatorname{Max} A$ on the Hilbert quantale $\mathscr{Q}(H)$ determined by any irreducible representation of the C*-algebra A on a Hilbert space H is indeed non-trivial. Indeed, in this case, each pure state of the Hilbert quantale $\mathscr{Q}(H)$ has direct image a pure state of the spectrum $\operatorname{Max} A$, in such a way that the direct image mapping then provides an embedding

$$\mathrm{P}(\varphi) : \mathrm{P}(\mathscr{Q}(H)) \to \mathrm{P}(\operatorname{Max} A)$$

of the set of pure states of the Hilbert quantale $\mathscr{Q}(H)$ in the set of pure states of the spectrum $\operatorname{Max} A$. Moreover, it may be seen by the preceding corollary that the image of this embedding then consists exactly of those pure states of $\operatorname{Max} A$ of which the inverse image is proper. In consequence, the representation is then non-trivial, since the set of pure states of the Hilbert quantale $\mathscr{Q}(H)$ is certainly non-empty by the fact that an irreducible representation is necessarily non-zero.

Explicitly, observing that each pure state of the Hilbert quantale $\mathscr{Q}(H)$ is obtained by choosing a non-zero element $x \in H$, recall that

$$\mathfrak{m}_x = \{a \in A | x\varphi_a = 0_H\}$$

is a maximal right ideal of the C*-algebra A, necessarily by its construction the direct image of the maximal right-sided element M_x of the Hilbert quantale $\mathscr{Q}(H)$ determined by the closed linear subspace of H generated by the element $x \in H$. By the irreducibility of the representation, the direct image $N_x\varphi_*$ of the pure state N_x corresponding to this maximal right-sided element M_x is necessarily a proper self-adjoint element of the spectrum $\operatorname{Max} A$ that contains the maximal right ideal $\mathfrak{m}_x$, hence by uniqueness is exactly the pure state

$$\mathfrak{n}_x = \mathfrak{m}_x \vee \mathfrak{m}_x^*$$

of the spectrum $\operatorname{Max} A$ associated with the maximal right ideal $\mathfrak{m}_x$. Hence, the direct image of the pure state $N_x \in \mathscr{Q}(H)$ of the Hilbert quantale $\mathscr{Q}(H)$ is indeed the pure state

$$\mathfrak{n}_x = N_x\varphi_*$$

of the spectrum $\operatorname{Max} A$. It is now asserted that the inverse image $\mathfrak{n}_x\varphi^*$ of this pure state $\mathfrak{n}_x$ is the pure state N_x of the Hilbert quantale $\mathscr{Q}(H)$.

To prove this, it suffices to show that the inverse image $\mathfrak{m}_x\varphi^*$ of the maximal right ideal $\mathfrak{m}_x$ is the maximal right-sided element $M_x \in \mathscr{Q}(H)$ of the Hilbert quantale $\mathscr{Q}(H)$.

Noting the definition of the element $M_x \in \mathcal{Q}(H)$ given earlier, observe immediately that

$$\mathfrak{m}_x\varphi^* \leq M_x,$$

since by the construction of the maximal right ideal $\mathfrak{m}_x$ the image of the linear subspace generated by the element $x \in H$ under the sup-preserving mapping $\mathfrak{m}_x\varphi^*$ is necessarily the zero subspace. To show conversely that

$$M_x \leq \mathfrak{m}_x\varphi^*,$$

it is enough to verify that the sup-preserving mapping $\mathfrak{m}_x\varphi^*$ maps the linear subspace generated by any element $y \in H$ that is linearly independent of the element $x \in H$ to the closed linear subspace H itself. However, in this situation, by a well-known construction [4], for any $z \in H$ there exists an element $a_z \in A$ of the C*-algebra A to which the representation assigns a bounded linear operator that maps $x \in H$ to zero and $y \in H$ to the given element $z \in H$. Since the first of these conditions states exactly that $a_z \in \mathfrak{m}_x$, allowing $z \in H$ to vary over the Hilbert space H ensures that the sup-preserving mapping given by $\bigvee_{z\in H}\varphi_{a_z}$, and hence a fortiori that given by $\bigvee_{a\in\mathfrak{m}_x}\varphi_a$, maps the linear subspace generated by $y \in H$ to the Hilbert space H itself. Thus, the inverse image $\mathfrak{m}_x\varphi^*$ of the maximal right ideal $\mathfrak{m}_x$ associated with the pure state $\mathfrak{n}_x$ is exactly the maximal right-sided element M_x associated with the pure state N_x. Hence, since the inverse image homomorphism preserves joins and involution, it follows that

$$\mathfrak{n}_x\varphi^* = (\mathfrak{m}_x \vee \mathfrak{m}_x^*)\varphi^* = \mathfrak{m}_x\,\varphi^* \vee (\mathfrak{m}_x\,\varphi^*)^* = M_x \vee M_x^* = N_x,$$

as asserted. To observe that the mapping

$$\mathrm{P}(\varphi) : \mathrm{P}(\mathcal{Q}(H)) \to \mathrm{P}(\mathrm{Max}\,A)$$

from the pure states of the Hilbert quantale $\mathcal{Q}(H)$ to the pure states of the spectrum $\mathrm{Max}\,A$ induced by the direct image mapping φ_* is indeed an embedding, we note simply that, by the above identity, the inverse image homomorphism provides a splitting for the direct image mapping on pure states, giving the required condition.

Indeed, these observations concerning the irreducible representation determined by an irreducible representation of a C*-algebra A on a Hilbert space H may be extended to the following specialisation of the preceding corollary:

Corollary 8.3. *For any irreducible representation*

$$\varphi : \mathrm{Max}\,A \to \mathcal{Q}(S)$$

of the spectrum of a C**-algebra A on an atomic orthocomplemented sup-lattice S, the following conditions on a pure state N of the Hilbert quantale $\mathcal{Q}(S)$ are equivalent*:

(a) *the pure state N is the inverse image of a pure state $\mathfrak{n}$ of the spectrum* $\mathrm{Max}\,A$;

(b) *the pure state N has direct image a pure state $\mathfrak{n}$ of the spectrum* $\mathrm{Max}\,A$ *of which the inverse image is the pure state N.*

Moreover, in the case that these equivalent conditions are satisfied by each pure state of the Hilbert quantale $\mathscr{Q}(S)$, *the direct image mapping provides an embedding*

$$\mathrm{P}(\varphi) : \mathrm{P}(\mathscr{Q}(H)) \to \mathrm{P}(\mathrm{Max}\, A)$$

of the pure states of the Hilbert quantale $\mathscr{Q}(S)$ *in the pure states of the spectrum* $\mathrm{Max}\, A$.

Proof. Evidently, the first condition is implied by the second. Suppose conversely that the pure state N of the Hilbert quantale $\mathscr{Q}(S)$ is the inverse image $\mathfrak{n}\varphi^*$ of a pure state $\mathfrak{n}$ of the spectrum $\mathrm{Max}\, A$. It is asserted that the direct image $N\varphi_*$ of N is exactly $\mathfrak{n}$, hence that $N\varphi_*\varphi^* = N$. For, observe firstly that, by the adjointness of the inverse image and direct image mappings, we have that $\mathfrak{n}\varphi^* \le N$ implies that $\mathfrak{n} \le N\varphi_*$. However, since the element $N\varphi_*$ is self-adjoint, by the remarks preceding the corollary above, and proper, by the irreducibility of the representation, in the spectrum $\mathrm{Max}\, A$, this necessarily implies that $N\varphi_* = \mathfrak{n}$, by the maximality of the pure state $\mathfrak{n}$. Hence, the first condition implies the second, giving the required equivalence.

Suppose now that these equivalent conditions are satisfied by each pure state of the Hilbert quantale $\mathscr{Q}(S)$. Then the direct image mapping evidently provides a mapping

$$\mathrm{P}(\varphi) : \mathrm{P}(\mathscr{Q}(H)) \to \mathrm{P}(\mathrm{Max}\, A)$$

from the pure states of the Hilbert quantale $\mathscr{Q}(S)$ to the pure states of the spectrum $\mathrm{Max}\, A$. That this determines an embedding of the pure states of $\mathscr{Q}(S)$ amongst such pure states of $\mathrm{Max}\, A$ follows by noting that $N\varphi_* = N'\varphi_*$ implies that $N = N\varphi_*\varphi^* = N'\varphi_*\varphi^* = N'$, by the above remarks. Indeed, it may be noted by the preceding corollary that a pure state $\mathfrak{n}$ of $\mathrm{Max}\, A$ lies in the image of this embedding exactly if its inverse image $\mathfrak{n}\varphi^*$ is proper, hence by the above remarks necessarily pure. In particular, the condition that the direct image mapping providing an embedding of the pure states N of the Hilbert quantale $\mathscr{Q}(S)$ in the pure states $\mathfrak{n}$ of the spectrum $\mathrm{Max}\, A$ may be seen to be equivalent to the requirement that each pure state N of the Hilbert quantale $\mathscr{Q}(S)$ satisfies the equivalent conditions of the corollary. □

It will later be seen that this condition on an irreducible representation

$$\varphi : \mathrm{Max}\, A \to \mathscr{Q}(S)$$

of the spectrum $\mathrm{Max}\, A$ on an atomic orthocomplemented sup-lattice S, that the direct image mapping provides an embedding

$$\mathrm{P}(\varphi) : \mathrm{P}(\mathscr{Q}(H)) \to \mathrm{P}(\mathrm{Max}\, A)$$

of the pure states $N \in \mathscr{Q}(S)$ of the Hilbert quantale $\mathscr{Q}(S)$ in the pure states of the spectrum $\mathrm{Max}\, A$, not only is satisfied by the irreducible representation

$$\varphi : \mathrm{Max}\, A \to \mathscr{Q}(H)$$

determined by an irreducible representation of the C*-algebra A on a Hilbert space H, but actually provides a characterisation of those irreducible representations of the spectrum $\mathrm{Max}\,A$ on an atomic orthocomplemented sup-lattice S that are equivalent to an irreducible representation of this kind, in the sense that will now be described.

9. The points of Max A

Throughout, the aim has been to arrive at a formulation of the concept of a point of the spectrum $\mathrm{Max}\,A$ of a C*-algebra A. Consideration of the spectrum $\mathrm{Max}\,A$ as the Lindenbaum algebra of propositions in a theory that generalises that considered constructively in the classical commutative case, and, more specifically, intuitive expectations concerning the nature of points of a quantale of the kind of $\mathrm{Max}\,A$, have led us to consider representations

$$\varphi : \mathrm{Max}\,A \to \mathcal{Q}(S)$$

of the spectrum on atomic orthocomplemented sup-lattices S. Amongst these are those representations that arise from an irreducible representation

$$\varphi : A \to \mathcal{B}(H)$$

of the C*-algebra A on a Hilbert space H. Extending to this context the concept of equivalence of representations of a C*-algebra A, we have the following definition:

Definition 9.1. By an *equivalence* of representations

$$\varphi : \mathrm{Max}\,A \to \mathcal{Q}(S), \quad \varphi' : \mathrm{Max}\,A' \to \mathcal{Q}(S')$$

of the spectrum $\mathrm{Max}\,A$ of a C*-algebra A on orthocomplemented sup-lattices S, S' will be meant an isomorphism

$$\eta : S \to S'$$

of orthocomplemented sup-lattices for which the corresponding isomorphism

$$\sigma_\eta : \mathcal{Q}(S) \to \mathcal{Q}(S')$$

makes the diagram

$$\begin{tikzcd} \mathrm{Max}\,A \arrow[r, "\varphi"] \arrow[dr, "\varphi'"'] & \mathcal{Q}(S) \arrow[d, "\sigma_\eta"] \\ & \mathcal{Q}(S') \end{tikzcd}$$

commute.

It may be proved straightforwardly, applying the remarks made earlier, that representations

$$\varphi : A \to \mathcal{B}(H), \quad \varphi' : A \to \mathcal{B}(H')$$

of a C*-algebra A on Hilbert spaces H, H' are equivalent exactly if the corresponding representations

$$\varphi_{\mathrm{Max}\,A} : \mathrm{Max}\,A \to \mathscr{Q}(H), \quad \varphi'_{\mathrm{Max}\,A} : \mathrm{Max}\,A \to \mathscr{Q}(H')$$

of the spectrum Max A are equivalent in the present sense.

With this in mind, for the sake of brevity we give the following definition.

Definition 9.2. By a *Hilbert representation* of the spectrum Max A of a C*-algebra A will be meant a representation

$$\varphi : \mathrm{Max}\,A \to \mathscr{Q}(S)$$

which is equivalent to the representation of Max A arising from a representation of the C*-algebra A on a Hilbert space H.

At this point, it should be remarked that the concept of a Hilbert representation being irreducible is that which is referred to classically as topological irreducibility, being based on the requirement that there exists no non-trivial *closed* subspace of the Hilbert space H that is invariant under the action of the representation of the C*-algebra A. In the present situation, this has been abstracted by considering the orthocomplemented sup-lattice $\mathscr{P}(H)$ of closed linear subspaces of the Hilbert space H. It is well-known that in the context of representations of a C*-algebra A on a Hilbert space H the concept of topological irreducibility is equivalent, albeit non-trivially, to that of algebraic irreducibility, that is that there exists no non-trivial subspace of H that is invariant under the action of the C*-algebra A.

In the present context, even within the Hilbert quantale of the atomic orthocomplemented sup-lattice P(H) of *closed* linear subspaces of H, this concept may be recovered in the following form:

Definition 9.3. A representation

$$\varphi : \mathrm{Max}\,A \to \mathscr{Q}(S)$$

of the spectrum of a C*-algebra A on an atomic orthocomplemented sup-lattice S will be said to be *algebraically irreducible* provided that it is non-zero, and that each atom $x \in S$ is a *cyclic generator* of the representation, in the sense that the subset

$$\{x\varphi_M \in S \mid M \in \mathrm{Max}\,A\}$$

is exactly the sup-lattice S.

It may be remarked that the equivalence of this classically with the topological irreducibility of the representation establishes the fact that every irreducible representation of the C*-algebra A on Hilbert space H arises from a pure state of the C*-algebra A. In the case of an irreducible representation of the spectrum Max A on an atomic orthocomplemented sup-lattice S, the concept of algebraic irreducibility provides the basis for a characterisation of those irreducible representations of the Gelfand quantale Max A

that are equivalent to those obtained from irreducible representations of the C*-algebra A in the classical sense, thereby taking a first step towards identifying the concept of a point of the spectrum $\mathrm{Max}\,A$.

Theorem 9.1. *For any* C**-algebra* A, *an irreducible representation*

$$\varphi : \mathrm{Max}\,A \to \mathscr{Q}(S)$$

of the spectrum $\mathrm{Max}\,A$ *on an atomic orthocomplemented sup-lattice* S *is a Hilbert representation if, and only if, it is non-trivial and algebraically irreducible.*

Proof. That the condition is necessary may be remarked by observing that we have already shown that the irreducible representation

$$\varphi : \mathrm{Max}\,A \to \mathscr{Q}(H)$$

of the spectrum $\mathrm{Max}\,A$ determined canonically by an irreducible representation of the C*-algebra A on a Hilbert space H is necessarily non-trivial, indeed actually determines an embedding of the pure states of the Hilbert quantale $\mathscr{Q}(H)$ in the pure states of the spectrum $\mathrm{Max}\,A$. That the irreducible representation is algebraically irreducible in the present sense is just a restatement within the context of quantales of the observation that any irreducible representation of a C*-algebra A on a Hilbert space H is indeed algebraically irreducible in the classical sense. The irreducible representation

$$\varphi : \mathrm{Max}\,A \to \mathscr{Q}(H)$$

is thus indeed non-trivial and algebraically irreducible, as asserted.

It remains to prove that the condition is sufficient, in other words that with a non-trivial algebraically irreducible representation

$$\varphi : \mathrm{Max}\,A \to \mathscr{Q}(S)$$

of the spectrum $\mathrm{Max}\,A$ of a C*-algebra A on an atomic orthocomplemented sup-lattice S may be associated a Hilbert space H together with an isomorphism

$$\eta : \mathscr{P}(H) \to S$$

of orthocomplemented sup-lattices from the lattice of closed linear subspaces of the Hilbert space H to the orthocomplemented sup-lattice S, inducing an equivalence of representations of the Gelfand quantale $\mathrm{Max}\,A$. It is with establishing this that the remainder of the proof will be concerned.

It may be remarked at this point that we actually prove a slightly sharper result, namely that the irreducible representation is a Hilbert representation provided that there is a pure state $N_x \in \mathscr{Q}(S)$ of the Hilbert quantale $\mathscr{Q}(S)$ of which the direct image $N_x\varphi_* \in \mathrm{Max}\,A$ is a pure state of the spectrum $\mathrm{Max}\,A$, and for which the corresponding atom $x \in S$ is a cyclic generator of the representation, a situation that we shall describe by saying that the pure state $N_x \in \mathscr{Q}(S)$ *generates* the representation. For, evidently, supposing that

$$\varphi : \mathrm{Max}\,A \to \mathscr{Q}(S)$$

is a non-trivial algebraically irreducible representation of the spectrum $\mathrm{Max}\,A$ of the C*-algebra A on the atomic orthocomplemented sup-lattice S, then by the non-triviality of the irreducible representation, there exists a pure state $\mathfrak{n}$ of the spectrum $\mathrm{Max}\,A$ of which the inverse image is a proper element of the Hilbert quantale $\mathscr{Q}(S)$. By Corollary 8.2, there exists a pure state N of the Hilbert quantale $\mathscr{Q}(S)$ of which the direct image $N\varphi_*$ is necessarily the pure state $\mathfrak{n}$. Denoting by $x \in S$ the atom of the atomic orthocomplemented sup-lattice S for which N is the pure state $N_x \in \mathscr{Q}(S)$, one observes by the proof of Corollary 8.2 that the maximal right ideal $\mathfrak{m}$ of the C*-algebra A associated with the pure state $\mathfrak{n}$ is equally necessarily the direct image $M_x\varphi_*$ of the maximal right-sided element $M_x \in \mathscr{Q}(S)$ associated with the pure state $N_x \in \mathscr{Q}(S)$. Throughout the remainder of the proof, the direct images $M_x\varphi_*, N_x\varphi_*$ of these elements $M_x, N_x \in \mathscr{Q}(S)$ will be denoted by $\mathfrak{m}_x$, $\mathfrak{n}_x$ respectively.

With this notation, it may be recalled that the quotient space $A/\mathfrak{m}_x$ of the C*-algebra A determined by the maximal right ideal $\mathfrak{m}_x$ is then a Hilbert space with respect to the inner product determined by the pure state of the C*-algebra A of which the kernel is the self-adjoint closed linear subspace $\mathfrak{n}_x$. Hence, there is a canonical isomorphism

$$P(A/\mathfrak{m}_x) \to \uparrow\mathfrak{m}_x$$

of sup-lattices from the sup-lattice of closed linear subspaces of the Hilbert space $A/\mathfrak{m}_x$ thereby determined to the up-segment $\uparrow\mathfrak{m}_x$ of the maximal right ideal $\mathfrak{m}_x$, which is an isomorphism of orthocomplemented sup-lattices with respect to the orthocomplement on $\uparrow\mathfrak{m}_x$ defined by the orthogonality relation given by setting

$$M \perp N \quad \text{if, and only if,} \quad M \& N^* \le \mathfrak{n}_x$$

for each M, $N \ge \mathfrak{m}_x$ in the quantale $\mathrm{Max}\,A$.

Equally, it may be recalled that there is a canonical isomorphism

$$\uparrow M_x \to S$$

of sup-lattices from the up-segment of the maximal right-sided element M_x of the Hilbert quantale $\mathscr{Q}(S)$ to the sup-lattice S, obtained by assigning to each sup-preserving mapping $\tau \ge M_x$ its value $x\tau \in S$ at the given atom $x \in S$. Moreover, the up-segment $\uparrow M_x$ is canonically an orthocomplemented sup-lattice, and the canonical mapping

$$\uparrow M_x \to S$$

an isomorphism of orthocomplemented sup-lattices, with respect to the orthocomplement on $\uparrow M_x$ defined by the orthogonality relation given by setting

$$M \perp N \quad \text{if, and only if,} \quad M \& N^* \le N_x$$

for each M, $N \ge M_x$ in the quantale $\mathscr{Q}(S)$.

It may therefore be remarked that to obtain an isomorphism of orthocomplemented sup-lattices from the orthocomplemented sup-lattice $\mathscr{P}(A/\mathfrak{m}_x)$ of closed linear subspaces of the Hilbert space $A/\mathfrak{m}_x$ to the orthocomplemented sup-lattice S, it is enough to obtain an isomorphism of orthocomplemented sup-lattices from the up-segment $\uparrow\mathfrak{m}_x$ of the maximal right-sided element $\mathfrak{m}_x$ of the spectrum $\mathrm{Max}\,A$ to the up-segment $\uparrow M_x$ of the maximal right-sided element M_x of the Hilbert quantale $\mathscr{Q}(S)$. Consider then the mapping

$$\psi^* : \uparrow\mathfrak{m}_x \to \uparrow M_x$$

obtained from the inverse image mapping

$$\varphi^* : \operatorname{Max} A \to \mathscr{Q}(S)$$

by assigning to each closed linear subspace N of the C*-algebra A containing the maximal right ideal $\mathfrak{m}_x$ the element $N\varphi^* \vee M_x$ of the Hilbert quantale $\mathscr{Q}(S)$, together with the mapping

$$\psi_* : \uparrow M_x \to \uparrow\mathfrak{m}_x$$

obtained by restricting the direct image mapping

$$\varphi_* : \mathscr{Q}(S) \to \operatorname{Max} A$$

to the up-segment of the maximal right-sided element $M_x \in \mathscr{Q}(S)$, of which the values lie naturally in the up-segment of $\mathfrak{m}_x = M_x\varphi_*$. It may be noted that the mapping ψ^* is necessarily sup-preserving, while the mapping ψ_* is at least order-preserving. It may also be remarked in passing that a consequence of the irreducible representation being shown to be Hilbert will be that $N\varphi^* \geq M_x$ for each closed linear subspace $N \geq \mathfrak{m}_x$, from which it follows that the mapping ψ^* is in fact obtained similarly by restricting the inverse image mapping to the up-segment of the maximal right ideal $\mathfrak{m}_x$.

It is asserted that, with respect to the canonical orthocomplements on the sup-lattices concerned, the mapping

$$\psi^* : \uparrow\mathfrak{m}_x \to \uparrow M_x$$

is an isomorphism of orthocomplemented sup-lattices, with inverse the mapping

$$\psi_* : \uparrow M_x \to \uparrow\mathfrak{m}_x.$$

To establish this, we show firstly that the mappings concerned are inverse in one direction, namely that

$$\uparrow M_x \xrightarrow{\psi_*} \uparrow\mathfrak{m}_x \xrightarrow{\psi^*} \uparrow M_x$$

is the identity mapping on the sup-lattice $\uparrow M_x$. For, recall that the elements of $\uparrow M_x$ are the mappings $\tau_s : S \to S$ defined for each $s \in S$ by

$$t\tau_s = \begin{cases} 1_S & \text{unless} \\ s & t = x \\ 0_S & t = 0_S \end{cases}$$

by the form of the sup-preserving mappings in the up-segment of M_x noted above. Hence,

$$\tau_s\psi_* = \bigvee_{N\varphi^* \leq \tau_s} N = \bigvee_{\varphi_N \leq \tau_s} N = \bigvee_{x\varphi_N \leq s} N,$$

respectively by the definition of the direct image mapping φ_*, the denotation by φ_N of the image of a closed linear subspace N of the C*-algebra A under the representation of the spectrum on the orthocomplemented sup-lattice S, and the observation that lying in the down-segment of τ_s is equivalent to taking value at $x \in S$ in the down-segment of $s \in S$. Now, applying the sup-preserving mapping ψ^*, we obtain that

$$\tau_s\psi_*\psi^* = \bigvee_{x\varphi_N \leq s} N\psi^* = \bigvee_{x\varphi_N \leq s} \varphi_N \vee M_x.$$

However, since this lies in the up-segment of M_x, to verify that it equals τ_s it suffices to check that its value at $x \in S$ is indeed that of τ_s, namely $s \in S$. But, by the hypothesis that the pure state $N_x \in \mathcal{Q}(S)$ generates the representation, hence that the element $x \in S$ is a cyclic generator for the representation, the subset

$$\{x\varphi_N \in S \mid N \in \operatorname{Max} A\}$$

is necessarily equal to S, and hence there exists a closed linear subspace M of the C*-algebra A for which $x\varphi_M = s$. Hence,

$$x\tau_s\psi_*\psi^* = \bigvee_{x\varphi_N \leq s} x\varphi_N \vee xM_x = x\varphi_M \vee x\lambda_x = s,$$

respectively by the pointwise ordering on the sup-preserving mappings on S, the observation that the supremum is realised at the closed linear subspace M, and the fact that $x\lambda_x = 0_S$. Since this is equal to the value $x\tau_s = s$ of τ_s at $x \in S$, it follows that the mappings are equal, hence the required composite is the identity on $\uparrow M_x$, as asserted.

It is asserted next that the sup-preserving mapping

$$\psi^* : \uparrow\mathfrak{m}_x \to \uparrow M_x$$

both preserves and reflects orthogonality, with respect to the canonical orthocomplements on the sup-lattices concerned. By the remarks already made, the relation of orthogonality between closed linear subspaces containing $\mathfrak{m}_x$ may be expressed by

$$M \perp N \quad \text{if, and only if,} \quad M \& N^* \leq \mathfrak{n}_x$$

for any $M, N \geq \mathfrak{m}_x$, for $\mathfrak{n}_x$ the pure state of the spectrum $\operatorname{Max} A$ corresponding to the maximal right ideal $\mathfrak{m}_x$ of the C*-algebra A. Similarly, it has been remarked that in the Hilbert quantale $\mathcal{Q}(S)$ determined by the orthocomplemented sup-lattice S, the relation of orthogonality on the up-segment of the maximal right-sided element M_x may be expressed by

$$\tau_s \perp \tau_t \quad \text{if, and only if,} \quad \tau_s \& \tau_t^* \leq N_x$$

for any $\tau_s, \tau_t \geq M_x$, similarly for N_x, the pure state of the Hilbert quantale $\mathcal{Q}(S)$ corresponding to the maximal right-sided element M_x.

With these remarks, it follows immediately that the mapping

$$\psi^* : \uparrow\mathfrak{m}_x \to \uparrow M_x$$

preserves orthogonality. For, since $\mathfrak{m}_x\varphi^* \leq M_x$, by the construction of $\mathfrak{m}_x$, one has that $\mathfrak{m}_x^*\varphi^* \leq M_x^*$, by the fact that the inverse image mapping is involutive. Hence, on taking joins, we find that, by the construction of $\mathfrak{n}_x$ and by the observation concerning N_x, we have that $\mathfrak{n}_x\varphi^* \le N_x$. Hence,

$$\mathfrak{n}_x\varphi^* = (\mathfrak{m}_x \vee \mathfrak{m}_x^*)\varphi^* \le M_x \vee M_x^* = N_x.$$

But then, whenever $M, N \ge \mathfrak{m}_x$ are such that $M \perp N$ in the orthocomplemented sup-lattice $\uparrow\mathfrak{m}_x$, then $M\psi^* \perp N\psi^*$ in the orthocomplemented sup-lattice $\uparrow M_x$, since $M\&N^* \le \mathfrak{n}_x$ implies $(M\&N^*)\varphi^* \le N_x$, hence

$$\begin{aligned} M\psi^*\&(N\psi^*)^* &= (M\varphi^* \vee M_x)\&(N\varphi^* \vee M_x)^* \\ &= (\varphi_M \vee M_x)\&(\varphi_{N^*} \vee M_x^*) \\ &\le \varphi_M\&\varphi_{N^*} \vee (M_x \vee M_x^*) \\ &= (M\&N^*)\varphi^* \vee N_x \\ &= N_x, \end{aligned}$$

which establishes the required assertion.

For the converse assertion, suppose that closed linear subspaces M, N containing the maximal right ideal $\mathfrak{m}_x$ of the C*-algebra A for which $M\psi^* \perp N\psi^*$ in the orthocomplemented sup-lattice $\uparrow M_x$. By the observation that this means exactly that

$$M\psi^*\&(N\psi^*)^* \le N_x,$$

it follows by the definition of the mapping $\psi^* : \uparrow\mathfrak{m}_x \to \uparrow M_x$ that $\varphi_{M\&N^*} = \varphi_M\&\varphi_{N^*} \le (\varphi_M \vee M_x)\&(\varphi_{N^*} \vee M_x^*) = M\,\psi^*\&(N\,\psi^*)^* \le N_x$. By adjointness of the inverse and direct image mappings determined by the representation, it follows that $M\&N^* \le N_x\varphi_*$, and hence, by the irreducibility of the representation, that

$$M\&N^* \le \mathfrak{n}_x,$$

since the pure state N_x of the Hilbert quantale $\mathcal{Q}(S)$ has direct image the pure state $\mathfrak{n}_x$ of the spectrum $\operatorname{Max} A$. Hence, we have that $M \perp N$ in the orthocomplemented sup-lattice $\uparrow\mathfrak{m}_x$. The sup-preserving mapping

$$\psi^* : \uparrow\mathfrak{m}_x \to \uparrow M_x$$

therefore also reflects orthogonality.

It follows that the sup-preserving mapping

$$\psi^* : \uparrow\mathfrak{m}_x \to \uparrow M_x$$

is orthocomplement preserving, since for any closed linear subspace N containing $\mathfrak{m}_x$ one has that $N^\perp = \bigvee_{M \le N^\perp} M$. Hence, observing that any $\tau \ge M_x$ is necessarily of the form $M\psi^*$ by taking M to be $\tau\psi_*$, by the observation already made that ψ_* followed by ψ^* is the identity, one has that

$$(N\psi^*)^\perp = \bigvee_{\tau \le (N\psi^*)^\perp} \tau = \bigvee_{M\psi^* \le (N\psi^*)^\perp} M\psi^* = \bigvee_{M \le N^\perp} M\psi^* = N^\perp\psi^*,$$

since, by the above remarks, $M \le N^\perp$ if, and only if, $M\psi^* \le (N\psi^*)^\perp$.

With these remarks, we may conclude finally that

$$\psi^* : \uparrow\mathfrak{m}_x \to \uparrow M_x$$

is an isomorphism of orthocomplemented sup-lattices. The mapping concerned is already known to be an orthocomplement- and sup-preserving mapping from $\uparrow\mathfrak{m}_x$ onto $\uparrow M_x$. From the above remarks, it then follows that it is order-reflecting, since $M\,\psi^* \leq N\,\psi^*$ implies $M\,\psi^* \leq (N\psi^*)^{\perp\,\perp} = (N^{\perp}\psi^*)^{\perp}$ implies $M \leq N^{\perp\,\perp} = N$. The mapping is therefore also injective, hence an isomorphism of orthocomplemented sup-lattices, as asserted.

Observing that we already have canonical isomorphisms

$$\mathscr{P}(A/\mathfrak{m}_x) \to \uparrow\mathfrak{m}_x \quad \text{and} \quad \uparrow M_x \to S$$

of orthocomplemented sup-lattices, respectively by the mapping that identifies a closed linear subspace of the Hilbert space $A/\mathfrak{m}_x$ with a closed linear subspace lying above the maximal right ideal $\mathfrak{m}_x$ in the C*-algebra A, and the mapping that evaluates each $\tau \geq M_x$ at the arbitrarily chosen $x \in S$, this yields the desired isomorphism

$$\mathscr{P}(A/\mathfrak{m}_x) \to S$$

from the orthocomplemented sup-lattice of the Hilbert space $A/\mathfrak{m}_x$ determined by the maximal right ideal $\mathfrak{m}_x$ to the orthocomplemented sup-lattice S, from which it may be proved straightforwardly that the irreducible representation

$$\varphi : \operatorname{Max} A \to \mathscr{Q}(S)$$

on the Hilbert quantale $\mathscr{Q}(S)$ is indeed equivalent to that induced by the irreducible representation

$$A \to \mathscr{B}(A/\mathfrak{m}_x)$$

of the C*-algebra A determined by the maximal right ideal $\mathfrak{m}_x$, which completes the proof of the theorem. □

With the observation of the theorem, we have obtained a first characterisation, in terms of the concept of algebraic irreducibility, of those irreducible representations

$$\varphi : \operatorname{Max} A \to \mathscr{Q}(S)$$

of the spectrum of the C*-algebra A that correspond in some intuitive sense to its points. It has, however, been remarked that, in the case of a representation of the C*-algebra A on a Hilbert space H, the property of algebraic irreducibility is already implied by that of topological irreducibility, the notion of irreducibility considered here.

Having this classical equivalence in mind, we shall now observe that the equivalence that underlies this implication remains valid for any irreducible representation

$$\varphi : \operatorname{Max} A \to \mathscr{Q}(S)$$

that is both non-trivial and satisfies the conditions of the following:

Definition 9.4. A representation

$$\varphi : \mathrm{Max}\, A \to \mathcal{Q}(S)$$

of the spectrum of a C*-algebra A on an atomic orthocomplemented sup-lattice S will be said to be *regular* provided that it is non-zero and that for any distinct atoms $x, y \in S$ there exists an element $M \in \mathrm{Max}\, A$ for which

$$x\varphi_M = 0_S \quad \text{and} \quad y\varphi_M = 1_S.$$

It may first be noted that this condition on a representation already implies that the representation is irreducible. For, by the theorem of the preceding section, it suffices to show that the inverse image of the element $1_{\mathrm{Max}\, A}$ of the spectrum $\mathrm{Max}\, A$ is indeed the element $1_{\mathcal{Q}(S)}$ of the Hilbert quantale $\mathcal{Q}(S)$. For this to be the case, noting that this inverse image is exactly the element of $\mathcal{Q}(S)$ that we are denoting by $\varphi_A \in \mathcal{Q}(S)$, it is enough to show that

$$y\varphi_A = 1_S$$

for each atom $y \in S$. However, either the atomic orthocomplemented sup-lattice S has only a single atom, necessarily $1_S \in S$, in which case the assertion follows trivially since the inverse image homomorphism is unital, or, for each atom $y \in S$ we may choose an atom $x \in S$ distinct from $y \in S$ and apply the regularity of the representation to find an element $M \in \mathrm{Max}\, A$ for which $x\varphi_M = 0_S$ and $y\varphi_M = 1_S$. Noting that then $M \leq A$ implies that $1_S \leq y\varphi_M \leq y\varphi_A$, we have the required assertion. Hence, the representation

$$\varphi : \mathrm{Max}\, A \to \mathcal{Q}(S)$$

is irreducible. It may be seen straightforwardly that the irreducible representation

$$\varphi : \mathrm{Max}\, A \to \mathcal{Q}(H)$$

determined by an irreducible representation of the C*-algebra A on a Hilbert space H is indeed regular. For, by the transitivity theorem for irreducible representations of a C*-algebra A, to which reference has already been made [4], we have that for any linearly independent elements $x, y \in H$ there exists an element $a_z \in A$ to which the representation assigns a bounded linear operator that maps $x \in H$ to zero and $y \in H$ to any given element $z \in H$. Letting $M \in \mathrm{Max}\, A$ denote the closed linear subspace of A generated by these elements $a_z \in A$ for each $z \in H$, it follows that the inverse image $\varphi_M \in \mathcal{Q}(H)$ of this element of the spectrum $\mathrm{Max}\, A$ has the property that

$$\langle x\rangle\varphi_M = \langle 0\rangle \quad \text{and} \quad \langle y\rangle\varphi_M = H,$$

yielding the regularity of the representation.

The assertion that an irreducible representation that is regular is indeed algebraically irreducible is then the principal content of the following:

Corollary 9.2. *For any* C^**-algebra* A, *an irreducible representation*

$$\varphi : \mathrm{Max}\, A \to \mathcal{Q}(S)$$

of the spectrum Max A *on an atomic orthocomplemented sup-lattice* S *is a Hilbert representation if, and only if, it is non-trivial and regular.*

Proof. By the theorem already proved, it suffices to show that any regular representation

$$\varphi : \mathrm{Max}\, A \to \mathcal{Q}(S)$$

on an atomic orthocomplemented sup-lattice S is necessarily algebraically irreducible. Hence, given any atom $x \in S$ of the orthocomplemented sup-lattice S, we assert that $x \in S$ is indeed a cyclic generator of the representation. Recalling that this requires that the subset

$$\{x\varphi_M \in S \mid M \in \mathrm{Max}\, A\}$$

be shown to be exactly S, we note that by the atomicity of the sup-lattice S, it suffices to show that for each atom $y \in S$ there exists a closed linear subspace M of the C*-algebra A for which

$$x\varphi_M = y.$$

Consider then, for any given atom $y \in S$, the problem of finding such an element $M \in \mathrm{Max}\, A$. Evidently, one must search amongst those closed linear subspaces M for which

$$x\varphi_M \leq y.$$

Note that this condition is equivalent to requiring that

$$x\varphi_M M_y = 0,$$

by the definition of the maximal right-sided element M_y of the Hilbert quantale $\mathcal{Q}(S)$. However, this condition is in turn equivalent to the requirement that

$$\varphi_M \& M_y \leq M_x,$$

again by the definition of the maximal right-sided element M_x of $\mathcal{Q}(S)$.

Now, denoting correspondingly by $\mathfrak{m}_y$ the direct image $M_y\varphi_*$ of the maximal right-sided element M_y of the Hilbert quantale $\mathcal{Q}(S)$ determined by the given atom $y \in S$, consider the closed linear subspace L of the C*-algebra A given by

$$L = \mathfrak{m}_y^* \vee \mathfrak{m}_x.$$

Observe that this closed linear subspace lying above the maximal right-sided element $\mathfrak{m}_x$ is indeed proper, since otherwise $1_{\mathrm{Max}\, A} \leq \mathfrak{m}_y^* \vee \mathfrak{m}_x$ implies that $1_{\mathrm{Max}\, A}\varphi^* \leq (m_y\varphi^*)^* \vee (m_x\varphi^*) \leq M_y^* \vee M_x$ since, by adjointness, $\mathfrak{m}_y\varphi^* \leq M_y$ and $\mathfrak{m}_x\varphi^* \leq M_x$. However, $M_y^* \vee M_x$ is indeed proper, since, noting that $M_y^* = \lambda_y^* = \kappa_{y^\perp}$, it maps the element $x \in S$ to the element $y^\perp \in S$. Hence, in the orthocomplemented sup-lattice

$\uparrow\mathfrak{m}_x$, the proper element L has orthocomplement, obtained by taking the join of all those $M \geq \mathfrak{m}_x$ for which

$$L\&M^* \leq \mathfrak{n}_x,$$

necessarily non-zero, yielding the existence of a closed linear subspace M, strictly (by the properness of the element $L \in \uparrow\mathfrak{m}_x$) containing the maximal right ideal $\mathfrak{m}_x$, satisfying the above condition.

Concerning this closed linear subspace M, we note that

$$(m_y^* \vee m_x)\&M^* \leq \mathfrak{n}_x$$

implies that

$$\mathfrak{m}_y^*\&M^* \leq \mathfrak{n}_x,$$

since necessarily $\mathfrak{m}_x\&M^* \leq \mathfrak{m}_x \leq \mathfrak{n}_x$, by the right-sidedness of the element $\mathfrak{m}_x$. In turn, this is equivalent to the condition that

$$M\&\mathfrak{m}_y \leq \mathfrak{n}_x,$$

by the self-adjointness of the pure state $\mathfrak{n}_x$. Finally, by the right-sidedness of the element $\mathfrak{m}_y$, we have that

$$M\&\mathfrak{m}_y \leq m_x,$$

since $\mathfrak{m}_x$ is the largest right-sided element contained in the pure state $\mathfrak{n}_x$.

Applying the inverse image homomorphism to this inequality, we obtain that

$$\varphi_M\&\mathfrak{m}_y\varphi^* \leq \mathfrak{m}_x\varphi^*,$$

from which the required inequality

$$\varphi_M\&M_y \leq M_x$$

follows by observing that the inverse image of $\mathfrak{m}_y$ is exactly the element $M_y \in \mathscr{Q}(S)$. To see this, observe that by the regularity of the representation for each atom $z \in S$ distinct from $y \in S$ there exists an element $M \in \operatorname{Max} A$ for which $y\varphi_M = 0_S$ and $z\varphi_M = 1_S$. But, $y\varphi_M = 0_S$ implies that $\varphi_M \leq M_y$, hence by adjointness that $M \leq \mathfrak{m}_y$. Hence, we have that $\varphi_M \leq \mathfrak{m}_y\varphi^*$, giving that $z \in S$ is mapped by $\mathfrak{m}_y\varphi^*$ to $1_S \in S$. Since this holds for each atom $z \in S$ distinct from $y \in S$, it follows that the inverse image of $\mathfrak{m}_y \in \operatorname{Max} A$ is exactly $M_y \in \mathscr{Q}(S)$, as asserted. Noting that by adjointness one has that $\mathfrak{m}_x\varphi^* \leq M_x$, this yields the required inequality, hence, by the observation made above, that

$$x\varphi_M \leq y.$$

By the atomicity of the element $y \in S$, this implies that either $x\varphi_M = 0_S$ or $x\varphi_M = y$. However, $x\varphi_M = 0_S$ implies that

$$M\varphi^* \leq M_x,$$

and hence, by adjointness, that $M \leq M_x\varphi_* = \mathfrak{m}_x$, contradicting the observation that the closed linear subspace M lies strictly above the maximal right ideal $\mathfrak{m}_x$ in the spectrum Max A. Hence, there exists a closed linear subspace M such that

$$x\varphi_M = y,$$

as required, yielding that the element $x \in S$ is indeed a cyclic generator for the representation.

The representation is therefore algebraically irreducible, hence a Hilbert representation by the preceding theorem. $\square$

Finally, we may observe that the conditions for an irreducible representation to be a Hilbert representation may be expressed in terms which may be considered almost purely topological, in the sense that the condition requires merely that the direct image mapping behaves well on pure states.

Corollary 9.3. *For any* C*-*algebra A, an irreducible representation*

$$\varphi : \operatorname{Max} A \to \mathscr{Q}(S)$$

of the spectrum Max A *on an atomic orthocomplemented sup-lattice S is a Hilbert representation if, and only if, the direct image mapping canonically determines an embedding*

$$\mathrm{P}(\varphi) : \mathrm{P}(\mathscr{Q}(H)) \to \mathrm{P}(\operatorname{Max} A)$$

of the pure states of the Hilbert quantale $\mathscr{Q}(S)$ in the pure states of the spectrum Max A.

Proof. By the observations of the preceding section, it is known that the irreducible representation

$$\varphi : \operatorname{Max} A \to \mathscr{Q}(H)$$

determined by any irreducible representation of the C*-algebra A on a Hilbert space H canonically determines an embedding of pure states, from which the same is true of any equivalent representation. It remains therefore to show the converse, which we do by showing that any irreducible representation satisfying this condition is non-trivial and regular.

Suppose then that an irreducible representation

$$\varphi : \operatorname{Max} A \to \mathscr{Q}(S)$$

of the spectrum Max A on an atomic orthocomplemented sup-lattice S provides an embedding of the pure states of $\mathscr{Q}(S)$ in the pure states of Max A. Observe firstly that this implies that for each atom $x \in S$ the direct image $N_x\varphi_*$ of the pure state $N_x \in \mathscr{Q}(S)$ is a pure state of the spectrum Max A. Considering the direct image $M_x\varphi_*$ of the associated maximal right-sided element $M_x \in \mathscr{Q}(S)$, recall that by the proof of the theorem above $M_x\varphi_*$ is necessarily the maximal right ideal associated with the pure state $N_x\varphi_*$. Observing that by adjointness we have that

$$M_x\varphi_*\varphi^* \leq M_x,$$

we assert that this is in fact an equality. For suppose that

$$M_x\varphi_*\varphi^* \le M_y$$

for some other maximal right-sided element $M_y \in \mathcal{Q}(S)$. Then, by adjointness, it follows that $M_x\varphi_* \le M_y\varphi_*$, and hence that $M_x\varphi_* = M_y\varphi_*$, by the maximality of these right-sided elements of the spectrum $\operatorname{Max} A$. But then $N_x\varphi_* = N_y\varphi_*$, by the uniqueness of the pure state containing a given maximal right ideal of $\operatorname{Max} A$. By the condition that the direct image mapping is an embedding on pure states, it follows that $N_x = N_y$, and hence that $M_x = M_y$. Thus, M_x is the unique maximal right-sided element of $\mathcal{Q}(S)$ containing the right-sided element $M_x\varphi_*\varphi^* \in \mathcal{Q}(S)$. Hence,

$$M_x\varphi_*\varphi^* = M_x,$$

since each right-sided element of $\mathcal{Q}(S)$ is the meet of the maximal right-sided elements that contain it.

Since this holds for each atom $x \in S$, it follows that the irreducible representation is necessarily regular. For, given distinct atoms $x, y \in S$, we observe that $M_x\varphi_* \in \operatorname{Max} A$ is an element M of the spectrum $\operatorname{Max} A$ for which $x\varphi_M = 0_S$ and $y\varphi_M = 1_S$. For, by the notation for the action of the spectrum on the orthocomplemented sup-lattice S, φ_M denotes the inverse image of $M_x\varphi_*$, hence is exactly M_x, and, by the definition of the maximal right-sided element $M_x \in \mathcal{Q}(S)$, one has that $xM_x = 0_S$ and $yM_x = 1_S$, as required. The irreducible representation

$$\varphi : \operatorname{Max} A \to \mathcal{Q}(S)$$

is therefore regular, hence indeed a Hilbert representation, by the preceding corollary, giving the required equivalence. □

It may be remarked that this condition expresses exactly that pure states of the Hilbert quantale $\mathcal{Q}(S)$ should neither be allowed to be mapped to states that are no longer pure, nor be allowed to lose their identity by being merged with another pure state. It may be further remarked that Rosický has observed that an analogous characterisation may be obtained in terms of maximal right-sided elements of the Hilbert quantale.

Before considering the consequences of these results, we note one further aspect of the identification of irreducible representations of the C*-algebra A with the points of its spectrum $\operatorname{Max} A$, namely that the concept of a point of the spectrum nicely captures that of equivalence between irreducible representations, in the sense that any irreducible Hilbert representation of the spectrum of the C*-algebra A determines exactly those pure states of A that yield equivalent irreducible Hilbert representations of the C*-algebra, to which we shall refer as the pure states *associated* with the irreducible representation.

More explicitly, we have the following:

Corollary 9.4. *For any irreducible Hilbert representation*

$$\varphi : \operatorname{Max} A \to \mathcal{Q}(S)$$

of the spectrum $\mathrm{Max}\,A$ *of a* C^**-algebra* A *on an atomic orthocomplemented sup-lattice* S, *the pure states of* A *that are associated with the representation are each of the form* $\mathfrak{n}_x$ *for a unique atom* $x \in S$.

Proof. Firstly, we assert that if $\mathfrak{m}$ is any maximal right ideal of the C^*-algebra A for which the representation is equivalent to the canonical representation

$$\psi : \mathrm{Max}\,A \to \mathcal{Q}(A/\mathfrak{m})$$

on the Hilbert quantale $\mathcal{Q}(A/\mathfrak{m})$ determined by the Hilbert space $A/\mathfrak{m}$, then there exists an atom $x \in S$ of the orthocomplemented sup-lattice S that induces the given equivalence of representations. In other words, any irreducible representation of the C^*-algebra A that is equivalent to the given representation arises canonically from within the orthocomplemented sup-lattice S concerned.

For, suppose that

$$\eta : \mathcal{P}(A/\mathfrak{m}) \to S$$

is the isomorphism of orthocomplemented sup-lattices that yields the equivalence of representations, and let $x \in S$ denote the atom of S that is the image of the closed linear subspace of $A/\mathfrak{m}$ generated by the unit element $1_A \in A$ of the C^*-algebra A. Then, we assert that $\mathfrak{m}$ is the direct image in the spectrum $\mathrm{Max}\,A$ of the maximal right-sided element M_x of $\mathcal{Q}(S)$ determined by $x \in S$.

Evidently, since $\mathfrak{m}$ is a maximal right ideal of the C^*-algebra A, it suffices to show that $\mathfrak{m} \leq M_x\varphi_*$, which is equivalent to proving that $\mathfrak{m}\varphi^* \leq M_x$. By the definition of the maximal right-sided element M_x, this in turn is equivalent to establishing that $x\varphi_a = 0_S$ for each $a \in \mathfrak{m}$, in which φ_a denotes the sup-preserving mapping on S given by applying the representation to the closed linear subspace of A generated by the element $a \in A$. Denoting similarly by ψ_a the sup-preserving mapping on $\mathcal{P}(A/\mathfrak{m})$ obtained by right multiplication by $a \in A$ on the quotient space $A/\mathfrak{m}$, the condition for the equivalence of the representations yields that the diagram

$$\begin{array}{ccc}
\mathcal{P}(A/\mathfrak{m}) & \xrightarrow{\psi_a} & \mathcal{P}(A/\mathfrak{m}) \\
{\scriptstyle\eta}\big\downarrow & & \big\downarrow{\scriptstyle\eta} \\
S & \xrightarrow[\varphi_a]{} & S
\end{array}$$

commutes, and hence, in the evident notation, that $\overline{\langle 1_A\rangle}\eta\varphi_a = \overline{\langle 1_A\rangle}\psi_a\eta$. Observing that $\overline{\langle 1_A\rangle}\psi_a = \overline{\langle a\rangle} = 0_{\mathcal{P}(A/\mathfrak{m})}$ for $a \in \mathfrak{m}$, and that $x = \overline{\langle 1_A\rangle}\eta$ implies that $x\varphi_a = \overline{\langle 1_A\rangle}\eta\varphi_a$, it follows that $x\varphi_a = 0_S$ for each $a \in \mathfrak{m}$. Hence, $\mathfrak{m} \leq M_x\varphi_*$, from which equality follows by the maximality of the closed right ideal $\mathfrak{m}$.

The pure state $\mathfrak{n}$ determined by the maximal right ideal $\mathfrak{m}$ is therefore indeed obtained canonically from the irreducible Hilbert representation, as asserted. □

In particular, the pure states of the spectrum $\mathrm{Max}\,A$ that are associated with the irreducible representation are therefore exactly those of which the inverse image in the

Hilbert quantale $\mathcal{Q}(S)$ is proper, hence a pure state of $\mathcal{Q}(S)$. In this sense, an irreducible representation

$$\varphi : \mathrm{Max}\, A \to \mathcal{Q}(S)$$

of the spectrum of the C*-algebra A on an atomic orthocomplemented sup-lattice S, satisfying the equivalent characterisations established above, may be considered to determine exactly an equivalence class of irreducible representations of the C*-algebra A on Hilbert space in the classical sense, thereby identifying an appropriate concept of point for the Gelfand quantale $\mathrm{Max}\, A$.

10. Quantisation of points

This paper has been concerned with identifying a concept of point that is appropriate to the context of Gelfand quantales, which we view as a non-commutative generalisation of that of locales, hence as abstract non-commutative spaces. Evidently, one aspect of that programme is ultimately to provide a satisfactory definition of exactly what should be meant by a non-commutative topological space, in other words to identify those Gelfand quantales that are to be considered to be spatial. The viewpoint from which we commenced was that the spectrum $\mathrm{Max}\, A$ of a C*-algebra A is indeed likely to be a, not necessarily commutative, topological space in this sense. Moreover, that the points, in an appropriate non-commutative sense, of that spectrum $\mathrm{Max}\, A$ may be expected to be the equivalence classes of irreducible representations of the C*-algebra A on Hilbert space.

The approach that we have taken has been to observe that the spectrum $\mathrm{Max}\, A$ may, even in this non-commutative context, be constructed by taking the Lindenbaum algebra of a propositional theory $\mathbb{M}\mathrm{ax}\, A$ with which we are familiar in the commutative case. In the commutative case, the points of the spectrum $\mathrm{Max}\, A$, then within the context of locales, are exactly the classical models of the theory $\mathbb{M}\mathrm{ax}\, A$, in the sense of the models obtained by validation of the theory in the locale Ω, which is the topology of the one-point topological space $\mathbf{1}$. In turn, these are exactly the maps

$$\Phi : \mathbf{1} \to \mathrm{Max}\, A$$

of locales from the one-point space $\mathbf{1}$ to the spectrum $\mathrm{Max}\, A$, represented by the inverse image homomorphisms

$$\varphi : \mathrm{Max}\, A \to \Omega$$

to the locale Ω, which are more appropriate in this logical context. In particular, the points of the spectrum $\mathrm{Max}\, A$ coincide with the multiplicative linear functionals on the C*-algebra A, hence, equivalently, the maximal ideals of the C*-algebra A.

Consideration of the way in which an irreducible representation of a C*-algebra A on a Hilbert space H gives rise to a model of the theory $\mathbb{M}\mathrm{ax}\, A$ of its spectrum leads to the introduction of the concept of a Hilbert quantale, already met with elsewhere in providing a quantisation of the calculus of relations. These Hilbert quantales, obtained by taking the quantale of sup-preserving mappings on any orthocomplemented sup-lattice, provide a natural non-commutative generalisation of the locale Ω, in the sense that the localic reflection of any Hilbert quantale is exactly this locale. An irreducible representation of a C*-algebra A on a Hilbert space H has been seen to lead to a model of the theory $\mathbb{M}\mathrm{ax}\, A$ in the Hilbert quantale $\mathscr{Q}(H)$ determined by the orthocomplemented sup-lattice $\mathscr{P}(H)$ of closed linear subspaces of the Hilbert space H, a model which may therefore be considered classical in the sense that it is validated in this non-commutative abstraction of the locale Ω.

Working once again with the inverse image homomorphisms, rather than the maps of Gelfand quantales to which we shall return presently, the concept of a classical model of the theory $\mathrm{Max}\, A$ has therefore been abstracted to that of an irreducible representation

$$\varphi : \mathrm{Max}\, A \to \mathscr{Q}(S)$$

of the spectrum $\mathrm{Max}\, A$ of the C*-algebra A on the Hilbert quantale $\mathscr{Q}(S)$ of an orthocomplemented sup-lattice S, now supposed additionally, both to develop further the classical analogy, and to allow the required characterisation to be established, to be atomic. Within this context, the theorem that we have proved has characterised those irreducible representations

$$\varphi : \mathrm{Max}\, A \to \mathscr{Q}(S)$$

of the spectrum $\mathrm{Max}\, A$ of the C*-algebra A on atomic orthocomplemented sup-lattices S that indeed correspond to equivalence classes of irreducible representations of A on Hilbert space as being non-trivial and algebraically irreducible.

Placing to one side for a moment the question of non-triviality, the conclusion that we wish to draw is that the concept of a point of a Gelfand quantale Q is captured by that of an algebraically irreducible representation

$$\varphi : Q \to \mathscr{Q}(S)$$

of the Gelfand quantale Q on an atomic orthocomplemented sup-lattice S. In this view, the place of the Hilbert quantale $\mathscr{Q}(S)$ of an atomic orthocomplemented sup-lattice S is seen as that of the topology of a, now non-commutative, one-point space $\mathbf{1}_S$. That this standpoint is justified is evidenced by the observation that such a one-point space $\mathbf{1}_S$ indeed has exactly a single point:

Theorem 10.1. *For any atomic orthocomplemented sup-lattice S, any algebraically irreducible representation*

$$\varphi : \mathscr{Q}(S) \to \mathscr{Q}(T)$$

of the Hilbert quantale $\mathscr{Q}(S)$ on an atomic orthocomplemented sup-lattice T is canonically equivalent to the identity representation of $\mathscr{Q}(S)$ on the orthocomplemented sup-lattice S.

Proof. It must be shown that there exists an isomorphism

$$\eta : S \to T$$

of orthocomplemented sup-lattices which establishes an equivalence of representations between the identity representation of the Hilbert quantale $\mathcal{Q}(S)$ on the orthocomplemented sup-lattice S and the representation

$$\varphi : \mathcal{Q}(S) \to \mathcal{Q}(T),$$

in other words, that for each $\alpha \in \mathcal{Q}(S)$ the diagram

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[d, "\eta"'] & S \arrow[d, "\eta"] \\ T \arrow[r, "\alpha\varphi^*"'] & T \end{tikzcd}$$

commutes. Observe firstly that the representation is necessarily non-trivial. In fact, any pure state of the Hilbert quantale $\mathcal{Q}(S)$ has inverse image a proper element of the Hilbert quantale $\mathcal{Q}(T)$. To prove this, we observe that since the image of the homomorphism

$$\varphi : \mathcal{Q}(S) \to \mathcal{Q}(T)$$

is necessarily a Gelfand quantale, we may apply a result of Pelletier and Rosický [12] to observe that the homomorphism is necessarily an embedding. Hence, in particular, the inverse image of any proper element of $\mathcal{Q}(S)$ is necessarily a proper element of $\mathcal{Q}(T)$. Thus, any pure state of the Hilbert quantale $\mathcal{Q}(S)$ maps properly under the inverse image homomorphism of the representation. In particular, the representation is non-trivial.

Noting that the proof of Corollary 8.2 applies equally to the case of an irreducible representation

$$\varphi : \mathcal{Q}(S) \to \mathcal{Q}(T)$$

of the Hilbert quantale $\mathcal{Q}(S)$ on the atomic orthocomplemented sup-lattice T, it follows that each pure state of the Hilbert quantale $\mathcal{Q}(S)$ is the direct image of a pure state of the Hilbert quantale $\mathcal{Q}(T)$. Choosing arbitrarily a pure state N_x, say, of the Hilbert quantale $\mathcal{Q}(S)$, we may find a pure state N_y of the Hilbert quantale $\mathcal{Q}(T)$ of which the direct image $N_y\varphi_*$ is the pure state N_x of the Hilbert quantale $\mathcal{Q}(S)$. We assert that the maximal right-sided element M_y of $\mathcal{Q}(T)$ determined by the pure state N_y necessarily has direct image $M_y\varphi_*$ given by the maximal right-sided element M_x determined by the pure state N_x of the Hilbert quantale $\mathcal{Q}(S)$. For, since $M_x \leq N_x$ implies $M_x\varphi^* \leq N_x\varphi^*$, and $N_x \leq N_y\varphi_*$ implies adjointly that $N_x\varphi^* \leq N_y$, we have that $M_x\varphi^* \leq N_y$. Hence, since by the irreducibility of the representation one has that $M_x\varphi^*$ is right sided, we have that $M_x\varphi^* \leq M_y$, which is the largest right-sided element contained in the pure state N_y. Hence, adjointly, we have that $M_x \leq M_y\varphi_*$, from which we may deduce that the right-sided (by irreducibility) element $M_y\varphi_*$ of the Hilbert quantale $\mathcal{Q}(T)$ satisfies $M_x \leq M_y\varphi_* \leq N_y\varphi_* = N_x$. By the maximality of the right-sided element M_x within the pure state N_x it follows that $M_x = M_y\varphi_*$, as asserted.

Now, applying exactly the reasoning used in establishing the corresponding isomorphism of orthocomplemented sup-lattices needed to characterise those irreducible representations of the spectrum of a C*-algebra that are equivalent to representations on Hilbert space, it may be proved that the mapping

$$\uparrow M_x \rightarrow \uparrow M_y$$

obtained by assigning to each $\alpha \geq M_x$ the element $\alpha\varphi^* \vee M_y \geq M_y$ is an isomorphism of orthocomplemented sup-lattices. That the arguments used in that situation apply also in the present case may be seen by verifying that they depend only on the observation that the spectrum is a Gelfand quantale in which
(a) any proper right-sided element is contained in a pure state;
(b) any pure state $\mathfrak{n}$ canonically determines an orthocomplement by writing

$$M \perp N \quad \text{if, and only if,} \quad M \& N^* \leq \mathfrak{n}.$$

It is straightforward to observe that these conditions are satisfied equally in the Hilbert quantale $\mathcal{Q}(S)$, from which the required result follows.

Observing that there exist canonical isomorphisms, again of orthocomplemented sup-lattices,

$$S \rightarrow \uparrow M_x \quad \text{and} \quad \uparrow M_y \rightarrow T,$$

defined respectively by mapping $s \in S$ to $\kappa_s \vee M_x \in \uparrow M_x$, and $\beta \in \uparrow M_y$ to $y\beta \in T$, we may now define an isomorphism

$$\eta : S \rightarrow T$$

of orthocomplemented sup-lattices by composition of these isomorphisms with the isomorphism

$$\uparrow M_x \rightarrow \uparrow M_y$$

defined above. Noting that $M_x \leq M_y\varphi_*$ implies adjointly that $M_x\varphi^* \leq M_y$, this isomorphism may be verified straightforwardly to map $s \in S$ to the element $y(\kappa_s\varphi^*) \in T$.

To prove that for any $\alpha \in \mathcal{Q}(S)$ the diagram

$$\begin{array}{ccc} S & \xrightarrow{\alpha} & S \\ \eta\downarrow & & \downarrow\eta \\ T & \xrightarrow[\alpha\varphi^*]{} & T \end{array}$$

commutes, observe that any $s \in S$ maps respectively to $s\alpha \in S$ and then $y(\kappa_{s\alpha}\varphi^*) \in T$, and to $y(\kappa_s\varphi^*) \in T$ and then $(y(\kappa_s\varphi^*))(\alpha\varphi^*)$. But $(y(\kappa_s\varphi^*))(\alpha\varphi^*) = y(\kappa_s\varphi^* \& \alpha\varphi^*) = y((\kappa_s \& \alpha)\varphi^*)$. However, recalling that, for any $s \in S$, the left-sided element $\kappa_s \in \mathcal{Q}(S)$ is defined by

$$p\kappa_s = \begin{cases} s & \text{unless} \\ 0_S & p = 0_S \end{cases}$$

for any $p \in S$, it follows that

$$p(\kappa_s \& \alpha) = \begin{cases} s\alpha & \text{unless} \\ 0_S & p = 0_S \end{cases}$$

for any $p \in S$, and hence that $\kappa_s \& \alpha$ is exactly $\kappa_{s\alpha}$. Hence, $(\kappa_s \& \alpha)\varphi^* = \kappa_{s\alpha}\varphi^*$, giving that $y(\kappa_s \& \alpha)\varphi^* = y(\kappa_{s\alpha}\varphi^*)$, as required. In particular, the algebraically irreducible representation

$$\varphi : \mathcal{Q}(S) \to \mathcal{Q}(T)$$

of the Hilbert quantale $\mathcal{Q}(S)$ on the orthocomplemented sup-lattice T is therefore equivalent to the identity representation of $\mathcal{Q}(S)$ on the orthocomplemented sup-lattice S.

Finally, it may be remarked that the equivalence thereby established is actually canonical, in the sense that the isomorphism

$$\eta : S \to T$$

defined above is independent of the particular pure state of $\mathcal{Q}(S)$ chosen mapping properly into the Hilbert quantale $\mathcal{Q}(T)$. Explicitly, the element $y(\kappa_s\varphi^*) \in T$ to which an element $s \in S$ is mapped is independent of the particular atom $y \in T$ corresponding to the pure state $N_y \in \mathcal{Q}(T)$, since the left-sided element κ_s of $\mathcal{Q}(S)$ is necessarily mapped by the inverse image homomorphism φ^* of the irreducible representation to a left-sided element κ_t of the Hilbert quantale $\mathcal{Q}(T)$. Since the atom $y \in T$ is, of course, non-zero, we have that $y(\kappa_s\varphi^*)$ is then exactly this element $t \in T$. Hence, the isomorphism

$$\eta : S \to T$$

is the mapping uniquely determined by requiring that

$$\kappa_s\varphi^* = \kappa_{s\eta}$$

for each $s \in S$, so depends only on the algebraically irreducible representation

$$\varphi : \mathcal{Q}(S) \to \mathcal{Q}(T).$$

The representation is therefore naturally equivalent to the identity representation of $\mathcal{Q}(S)$ on the atomic orthocomplemented sup-lattice S. □

It may be remarked immediately that the theorem ceases to be valid in the case that algebraic irreducibility is weakened to irreducibility. Indeed, the locale Ω obtained by taking the Hilbert quantale $\mathcal{Q}(\mathbf{2})$ of the two-chain $\mathbf{2}$, hence the topology of the classical one-point space $\mathbf{1}$, admits a unique irreducible representation

$$\Omega \to \mathcal{Q}(T)$$

on any non-zero atomic orthocomplemented sup-lattice T. In accordance with the above theorem, this representation is algebraically irreducible exactly in the case that T is itself a two-chain. Moreover, it may be observed that the theorem may be sharpened into a necessary and sufficient condition by the following:

Corollary 10.2. *An irreducible representation*

$$\varphi : \mathcal{Q}(S) \to \mathcal{Q}(T)$$

of the Hilbert quantale $\mathcal{Q}(S)$ *of an atomic orthocomplemented sup-lattice* S *on an atomic orthocomplemented sup-lattice* T *is equivalent to the identity representation on the orthocomplemented sup-lattice* S *if, and only if, the representation is algebraically irreducible.*

Proof. The converse of the assertion of the theorem is obtained by observing that the identity representation on S, hence any representation equivalent to it, is necessarily algebraically irreducible, since, for any atoms $x, y \in S$, the sup-preserving mapping $\kappa_y \in \mathcal{Q}(S)$ defined by

$$s\kappa_y = \begin{cases} y & \text{unless} \\ 0_S & s = 0_S \end{cases}$$

for each $s \in S$ maps $x \in S$ to $y \in S$. □

It is therefore evident that the consideration of algebraically irreducible representations is intrinsic to the concept of point and of one-point space. With these observations in mind, we give the following definition:

Definition 10.1. By a *point* of a Gelfand quantale Q will be meant a map

$$\Phi : \mathbf{1}_S \to Q$$

of Gelfand quantales from the Hilbert quantale $\mathbf{1}_S$ of an atomic orthocomplemented sup-lattice S to the quantale Q, of which the inverse image homomorphism is an algebraically irreducible representation of the Gelfand quantale Q on the atomic orthocomplemented sup-lattice S.

Recalling that the orthocomplemented sup-lattice $\mathcal{P}(H)$ of closed linear subspaces of a Hilbert space H is necessarily atomic, the theorem characterising the irreducible representations of a C*-algebra A that are Hilbert may be interpreted as showing that a necessary and sufficient condition for an algebraically irreducible representation

$$\varphi : \operatorname{Max} A \to \mathcal{Q}(S)$$

of the spectrum Max A on an atomic orthocomplemented sup-lattice S to be Hilbert is that the representation is non-trivial. Having in mind the expectation that the points of the spectrum Max A of a C*-algebra A indeed correspond to the equivalence classes of irreducible representations of A on Hilbert space, we are led to formulate the following:

Conjecture 10.3. *Any algebraically irreducible representation*

$$\varphi : \operatorname{Max} A \to \mathcal{Q}(S)$$

of the spectrum Max A *of a* C*-*algebra* A *on an atomic orthocomplemented sup-lattice* S *is necessarily non-trivial.*

Provided that the conjecture can be proved, possibly by considering more carefully the concept of regularity or complete regularity satisfied by the Gelfand quantale Max A determined by any C*-algebra A, then one may have some confidence that the quantisation of the notion of point that we have sought to determine complies with at least some of the natural constraints that may be required.

Most important of these, since it has been our motivating example in the quantised case, is indeed that in the case of the spectrum of a C*-algebra A it yields equivalence classes of irreducible representations of A in the classical sense. But, equally importantly, this quantised requirement has been met with a concept of point that coincides in the case of a locale L, considered as a Gelfand quantale with respect to the involution given by the identity mapping, with the classical definition of a point as a homomorphism

$$\varphi : L \to \Omega$$

of locales from the locale L to the locale Ω, given by the two-chain, that is the topology of the classical one-point topological space $\mathbf{1}$. That is, the definition of point in the quantised situation is indeed an extension of the classical description from the category of locales to the category of Gelfand quantales within which it is canonically embedded as a full subcategory.

Explicitly, that this is indeed the case is because, in the case of a locale L, any irreducible representation

$$\varphi : L \to \mathcal{Q}(S)$$

on an atomic orthocomplemented sup-lattice necessarily maps each element of L into a two-sided element of $\mathcal{Q}(S)$, since each element of L is both left- and right-sided. However, the locale $\mathrm{I}(\mathcal{Q}(S))$ of two-sided elements of any Hilbert quantale $\mathcal{Q}(S)$ consists of exactly two elements, the zero and the top elements of the quantale $\mathcal{Q}(S)$, hence each element $a \in L$ is mapped by the representation either into $0_{\mathcal{Q}(S)}$ or into $1_{\mathcal{Q}(S)}$. Choosing any atom $x \in S$ of the atomic orthocomplemented sup-lattice S, observe that by the algebraic irreducibility of the representation there exists, in particular, an element $a \in L$ for which

$$x\varphi_a = x.$$

Evidently, this is not the case in the event that φ_a is the zero element of $\mathcal{Q}(S)$. Hence, it must be the case that φ_a is the top element $1_{\mathcal{Q}(S)} \in \mathcal{Q}(S)$. However, noting that the atom $x \in S$ is necessarily non-zero, it follows that $x\varphi_a \in S$, and hence $x \in S$ itself, is necessarily the top element $1_S \in S$. The atomic orthocomplemented sup-lattice S is therefore exactly the two-chain. In particular, the sup-lattice S is then the topology Ω of the classical one-point topological space $\mathbf{1}$, and the Hilbert quantale $\mathcal{Q}(S)$ is canonically isomorphic to Ω, as asserted. In other words, the points of a locale canonically considered as a Gelfand quantale are exactly its points in the classical sense.

To conclude, the ideas which have emerged throughout this investigation of the concept of point may be seen to relate closely to those introduced, perhaps on a more ad hoc basis, by Rosický. Recalling that the irreducible representations

$$\varphi : \mathrm{Max}\, A \to \mathscr{Q}(S)$$

on orthocomplemented sup-lattices are exactly those that restrict to a homomorphism of right-sided elements, we observe that the right side of any Gelfand quantale is a quantum frame in the sense [13] of Rosický. Indeed this, together with the need for a functorial description of the spectrum of a C*-algebra, was the motivation for the introduction of Gelfand quantales. Moreover, any orthocomplemented sup-lattice S may, as observed [13] by Rosický, be considered canonically a quantum frame, in which the product of the underlying quantale is given by that defined right-trivially, which is to say by

$$s\&t = \begin{cases} s & \text{unless} \\ 0_S & t = 0_S. \end{cases}$$

Indeed, the results that have been established above may be considered to place the insightful observations of Rosický in an intrinsically more natural context. The conjecture that is left to be decided requires that this work be extended to include an examination of the concept of regularity to the context of Gelfand quantales, in the expectation that its proof depends on a more carefully evolved description of the regularity enjoyed by the spectrum $\mathrm{Max}\, A$ of a non-commutative C*-algebra A.

Finally, there is a model-theoretic point to which we should return. In motivating the definition of the points of the spectrum $\mathrm{Max}\, A$ of a C*-algebra A, we have observed that the quantale $\mathrm{Max}\, A$ may be described in terms of a propositional theory $\mathbb{M}\mathrm{ax}\, A$ that is to be interpreted within a quantal logic. In particular, the concept of point emerged as a specialisation of the notion of a model of that theory, namely as an involutive homomorphism from the Lindenbaum algebra of the theory into a Gelfand quantale of the particular form $\mathscr{Q}(S)$. In the light of the view now taken of these Hilbert quantales determined by atomic orthocomplemented sup-lattices, as representing the quantised counterpart $\mathbf{1}_S$ of the one-point topological space $\mathbf{1}$, hence a multiplicative version of a quantum logic, it is interesting to revisit the concept of model determined in these cases. In the case of the homomorphism

$$\varphi : \mathrm{Max}\, A \to \mathscr{Q}(S)$$

determining a point of the spectrum $\mathrm{Max}\, A$, the corresponding model of the theory $\mathbb{M}\mathrm{ax}\, A$ assigns to each primitive proposition $a \in P$ of the theory a sup-preserving mapping

$$[\![a \in P]\!] : S \to S.$$

The interpretation that may be proposed of this assignment is that this truth value corresponds to the action of this proposition on the atomic orthocomplemented sup-lattice S that is induced by presenting the proposition, in some sense. In the case that the sup-lattice is indeed atomic, this action is, of course, completely determined by its effect on the atoms of the sup-lattice, which we shall refer to as its states. Recalling the convention by which we write the action of such a sup-preserving mapping to the right of the state to which it is applied, we note that the truth value of the proposition

$$a \in P \& b \in P,$$

for instance, is represented by the action on states of first presenting the proposition $a \in P$, and then the proposition $b \in P$. Similarly, the truth value of the disjunction $\bigvee a_i \in P$ of propositions $a_i \in P$ is represented by the action on states which assigns to a state what may be considered the subspace of the state space S spanned by the states obtained by presenting the propositions individually.

In this context, it is perhaps interesting to recall that the language within which the theory is described involves adjoining a proposition

$$a \in P^*$$

for each primitive proposition

$$a \in P$$

determined by an element of the C*-algebra A. Indeed, to any proposition of the theory an adjoint proposition is associated in this way. The axiomatisation of $\mathbb{M}\mathrm{ax}\, A$ then requires that the adjoint of this primitive proposition is provably equivalent to the proposition

$$a^* \in P$$

determined by the involute $a^* \in A$ of the element $a \in A$. The interpretation of this adjoint proposition $a \in P^*$ is by the involute in the Hilbert quantale $\mathcal{Q}(S)$ of the interpretation of the primitive proposition $a \in P$. In that the interpretation of the proposition $a \in P$ represents the action on states of the presentation of this proposition to the quantised system, that of $a \in P^*$ represents that of its presentation in reversed time. In terms of the interpretation discussed above, the sup-preserving mapping

$$[\![a \in P]\!]^* : S \to S$$

assigned to the involute of the proposition $a \in P$ is therefore the adjoint of that assigned to $a \in P$. It is interesting in this context to note that, from the definition of the involution in the Hilbert quantale $\mathcal{Q}(S)$, the action of this adjoint on elements of the orthocomplemented sup-lattice S is related to that of the sup-preserving mapping

$$[\![a \in P]\!] : S \to S$$

by the condition that

$$s[\![a \in P]\!]^* \perp t \quad \text{if, and only if,} \quad s \perp t[\![a \in P]\!]$$

for all $s, t \in S$.

In other words, the models obtained in this way are extremely reminiscent of interpretations of the propositions considered in laying the foundations of quantum mechanics [6]. Evidently, this is no surprise in the case of this particular theory, since C*-algebras are closely linked with quantum mechanics. However, it points a particular way forward as far as classical models of theories within quantal logic are concerned which may not initially have been anticipated. The conclusion has to be that, at least in these classical models, the interpretation of a proposition is by its action on the states of the system within which the theory is interpreted.

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